Mass matrix transforms in qubit field theory

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(Dated: November 2, 2007)

Circulant mass matrices for triples of charged and neutral leptons have been studied in the context of qubit quantum field theory. This note describes the discrete Fourier transform behind such matrices, and discusses a category theoretic interpretation of these operators.

PACS numbers: 03.67.-a, 03.67.Lx, 04.60.-m

INTRODUCTION

Using a measurement algebra approach to QFT, Brannen [1] recently recovered the Koide [2] formula

$$(\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau})^2=\frac{3}{2}(m_e+m_\mu+m_\tau) \tag{1}$$

for charged lepton masses in the form of a 3×3 circulant complex matrix, whose eigenvalues squared give the lepton masses to experimental precision. This analysis was extended to a set of three neutrinos, and the mass ratio predictions agree with preliminary neutrino oscillation data.

Here it is observed that the discrete Fourier transform [3] provides a further interpretation of the mass matrices, both as a duality between operators and eigenvalues and also as a link to the theory of quantum computation [4].

It is expected that other triples of Standard Model particles, namely baryons and mesons, will also be associated with 3×3 matrix operators of the same kind in accord with their preon structure [1] and the association of spatial directions to the number of particle generations, given by the three primitive idempotents of the measurement algebra.

FOURIER TRANSFORMS AND MASS MATRICES

A *circulant* matrix is built from its first row by adding cyclic permutations. In particular, a 3×3 circulant takes the form

$$\begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix} \tag{2}$$

where A, B and C will be complex numbers. Note that any such circulant is a combination of the three permutations (123), (231) and (312). For real eigenvalues λ_k it is essential that A be real and $C=\overline{B}$. Thus a mass matrix [1] takes the form

$$C=\eta\begin{pmatrix} 1 & re^{i\theta} & re^{-i\theta} \\ re^{-i\theta} & 1 & re^{i\theta} \\ re^{i\theta} & re^{-i\theta} & 1 \end{pmatrix} \tag{3}$$

for real η, r and θ. In terms of these parameters, the eigenvalues are given by

$$\lambda_k=\eta(1+2r\cos(\theta+\frac{2\pi k}{3}))$$

The Koide formula (1) follows when $r^2=\frac{1}{2}$ and this choice may be applied also to the neutrino matrix.

In the $n\times n$ case, the discrete Fourier transform [3][4] interchanges the set of eigenvalues λ_k (assumed distinct) and matrix entries $A_1,A_2,A_3,\cdots,A_n$ via

$$\lambda_k=\sum_j e^{\frac{2\pi ijk}{n}}A_j \tag{4}$$
$$A_j=\frac{1}{n}\sum_k e^{-\frac{2\pi ijk}{n}}\lambda_k$$

Viewing the eigenvalues as a diagonal matrix, the transform interchanges the bases of projection operators and cyclic permutations. For real eigenvalues (m_1,m_2,m_3) with $m_i=\lambda_i^2$ in the above, and letting $\omega=e^{\frac{2\pi i}{3}}$, the transform takes the diagonal matrix to the circulant

$$\begin{pmatrix} m_1+m_2+m_3 & m_1\omega+m_2\omega^2+m_3 & m_1\omega^2+m_2\omega+m_3 \\ m_1\omega^2+m_2\omega+m_3 & m_1+m_2+m_3 & m_1\omega+m_2\omega^2+m_3 \\ m_1\omega+m_2\omega^2+m_3 & m_1\omega^2+m_2\omega+m_3 & m_1+m_2+m_3 \end{pmatrix}$$

which must be the square of (3) since the square of a circulant is a circulant. Thus a choice of scale is specified by $\eta=\frac{1}{3}(m_1+m_2+m_3)$.

A 3×3 matrix is viewed as a function on the discrete torus $\mathbb{Z}_3\times\mathbb{Z}_3$, which has a quantum description in terms of the convolution product for matrices [3]. Letting $D_{ij}=\delta_{ij}\omega^i$ this product satisfies the Weyl rule

$$D*(312)=\omega(312)*D$$

where the phase $\frac{2\pi}{3}$ is proportional to $\hbar^{-1}$. This associates Planck's constant with a heirarchy $\mathbb{N}$ determined by the size of the matrix, but the continuum limit is obtained via $\hbar\to\infty$ rather than $\hbar\to 0$.

If masses are to be thought of as quantum numbers, then why are their values so awkward in comparison to, say, spin? For 2×2 circulants with entries A and B, the eigenvectors are $(1,1)$ and $(1,-1)$ with eigenvalues $(A+B)$ and $(A-B)$ respectively. For example, for the

Pauli swap matrix σ_x, with $A = 0$, the spin eigenvalues are ± 1. Complexity in the eigenvalue set only arises in dimension three or higher.

Degenerate eigenvalues $\frac{\lambda_k}{\eta} \in \{1-r, 1-r, 1+2r\}$ occur when $\theta = 0$ and all matrix entries are real. Although this pattern does not describe the leptons, we observe that it is the typical composition of masses for baryon constituents. Since such mass operators arise in a preon model that unifies particle structure, it is expected that all standard model bound states and resonances may be arranged into mass triples.

In quantum computation [4] a Fourier transform is also defined in this way, acting on a set of n basis states. An N qubit computer has $n = 2^N$ basis states. The transform is unitary because it may be built from unitary gates, namely the Hadamard gate $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$ and the series

$$B_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{pmatrix}$$

By analogy, a mass computation with 3^N basis states uses ternary digits, so the gates B_k would be replaced by gates

$$T_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3^k}} & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3^k}} \end{pmatrix} \quad (5)$$

which are also unitary. It is not clear, however, what operator should replace the Hadamard gate H, since it has no 3×3 analogue. Noting that σ_z carries the spin eigenvalues, one possibility is a combination of mass diagonals and the generator (312), but such an operator is *not* unitary. Such non-unitary processes would have major implications for the black hole paradox, which may only conserve qubit (flat space) information, but not gravitational information. On the other hand, the meaning of time itself is altered in this approach, which does not assume a globally defined time for a nonsensical universal observer.

Note the similarity between the T_k and powers of the diagonal D that appears in the Weyl relations. Given the direct application of the qubit Fourier transform to number factoring, this associates ternary factorisation with the quantum torus.

DISCUSSION

The mass matrices arise from a one dimensional discrete transform, which itself involves commutative variables. However, it is seen that phase space variables satisfy the Weyl algebra of the quantum plane. Is there a noncommutative transform that extends this analysis to nonclassical underlying spaces? This is relevant to the question of extending the perturbative rest mass computations [1] to a nonperturbative regime dealing with physical scale.

Kapranov [5] has recently considered path spaces approximated by cubical paths, each of which is represented by a noncommutative monomial in the spatial directions. In dimension $d > 1$ a noncommutative Fourier transform relates measures on the space of paths to functions of the noncommuting variables. The basic idea is that a path integral is just a map from a noncommutative ring to a suitable commutative subring. In this way, particle masses [1] could arise as path integral invariants.

Taking T-duality seriously, one also expects to deal with nonassociativity. From a category theoretic point of view, both noncommutative and nonassociative structures can be dealt with in a unified framework. The cohomological element of interest here is the parity cube axiom, which describes the now familiar pentagon law on five of its faces. In a sufficiently lax algebraic setting, such as for tetracategories, the sixth face may break this law, providing the deformation parameter that turns a pentagon into a hexagon representing the permutation group S_3 [6].

The generation count by primitive idempotents [1] is confirmed by the string theoretic index theorem argument applied to the Riemann moduli space of the six punctured sphere, which has an orbifold Euler characteristic [7] of -6. The six punctures are associated to the six faces of a cube via a dual vertex, which is thickened to a sphere. Note that cohomological integrals for such moduli spaces commonly appear in QFT computations as multiple zeta values and polylogarithms.

For helpful discussions I thank Carl Brannen, Michael Rios, Matti Pitkanen, Tony Smith and Louise Riofrio.

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