Gluon Phenomenology and a Linear Topos

by

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Disclaimer

I hereby declare that this thesis is entirely my own creation, based on work done with my collaborator Michael Rios and under the supervision of Dr William Joyce (chapters 5 and 6). No part of the thesis will be used towards a qualification at any other institution.

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But when I said that nothing had been done I erred in one important matter. We had definitely committed ourselves and were half-way out of our ruts. We had put down our passage money - booked a sailing to Bombay. This may sound too simple, but is great in consequence. Until one is committed, there is hesitancy, the chance to draw back, always ineffectiveness. Concerning all acts of initiative (and creation), there is one elementary truth, the ignorance of which kills countless ideas and splendid plans: that the moment one definitely commits oneself, then Providence moves too. All sorts of things occur to help one that would never otherwise have occurred. A whole stream of events issues from the decision, raising in one's favour all manner of unforeseen incidents and meetings and material assistance, which no man could have dreamt would have come his way.

W. H. Murray, *The Scottish Himalayan Expedition*, 1951

Contents

Abstract

In thinking about quantum causality one would like to approach rigorous QFT from outside the perspective of QFT, which one expects to recover only in a specific physical domain of quantum gravity. This thesis considers issues in causality using Category Theory, and their application to field theoretic observables. It appears that an abstract categorical Machian principle of duality for a ribbon graph calculus has the potential to incorporate the recent calculation of particle rest masses by Brannen [Bra], as well as the Bilson-Thompson characterisation of the particles of the Standard Model [BT].

This thesis shows how Veneziano n-point functions may be recovered in such a framework, using cohomological techniques inspired by twistor theory and recent MHV techniques introduced in [Wit04a]. This distinct approach fits into a rich framework of higher operads [Bata], leaving room for a generalisation to other physical amplitudes.

The utility of operads raises the question of a categorical description for the underlying physical logic. We need to consider quantum analogues of a *topos*. Grothendieck's concept of a topos [Gro] is a genuine extension of the notion of a space that incorporates a logic internal to itself. Conventional quantum logic has yet to be put into a form of equal utility, although its logic has been formulated in category theoretic terms [Coe].

Axioms for a quantum topos are given in this thesis, in terms of braided monoidal categories. The associated logic is analysed and, in particular, elements of linear vector space logic are shown to be recovered. The usefulness of doing so for ordinary quantum computation was made apparent recently by Coecke et al [Pava]. Vector spaces underly every notion of algebra, and a new perspective on it is therefore useful. The concept of state vector is also readdressed in the language of tricategories.

1 Introduction

> *"It is surprising to be told that a* trivial *system suffers from intractable infinities."*
>
> E. Witten [Wit88]

Physical theory is currently undergoing a revolution of thought on a par with that of the Copernican revolution. For a long time the incompatibility of the successful Standard Model and the theory of General Relativity has generated countless attempts at modifying the principles of one or the other, or both. Now experimental anomalies in the standard picture have forced us to search even further for completely new fundamental principles. On the mathematical side, real progress has been made on formulating a rigorous language for quantum field theory. But how do such techniques arise in a quantum gravitational context? Without a clearer understanding of new physics, it is uncertain how even the most advanced techniques could be used to predict quantitative results for upcoming experiments such as the LHC.

Heisenberg pointed out that particles were not fundamental because every particle in some sense contained all others [ed.73]. In viewing particles as building block systems this is no doubt true. However, the particles of QFT do represent fundamental kinds of proposition. Thus logic and algebra, rather than algebra alone, are necessary for a clear description of both the Standard Model and the new theory of quantum gravity. In mathematics, only category theory can combine these two disciplines.

Category theory is also about geometry. The concept of *point*, or spacetime event, is greatly abstracted from the idea of a point as an element of the set $\mathbb{R}^n$, the model for manifolds. In quantum gravity one tends to view the classical spacetime as an emergent phenomenon arising from the combination of large numbers of basic gravitational states. Unfortunately, this loose viewpoint is borrowed from the meaning of observable in particle physics and, by itself, adds nothing to our understanding of mass.

On the face of it, rest mass is a classical concept, even as it characterises the inertia of a charged quantum particle as measured in a mass spectrometer, because the mass depends on the curvature of the particle's path as it travels at a known small velocity in a magnetic field. This requires a background template of rods and clocks to define a laboratory frame. However, mass can potentially be viewed as a quantum observable not unlike spin. Recall that the Stern-Gerlach experiment takes a beam of electrons and observes its twofold splitting through a magnetic field, defining spin. This beam conveniently contains only electrons, and not muons or tau particles. But we could imagine a beam of (spin up) electrons, muons and tau particles which is split threefold by the particle masses.

In his Clifford algebra density matrix approach to QFT, Brannen [Bra] has constructed mass operators whose eigenvalues yield the charged lepton masses to within experimental precision. These operators rely at present on a small splitting parameter whose theoretical origin is not yet understood, but they clearly indicate the potential for a first principles derivation of particle masses within that framework, which is very distinct from string theory or LQG or other popular approaches to gravity.

A test of any new approach is its ability to explain the recent successes of the MHV diagram technique and its spinoffs [Wit04a][Svr]. The physical relevance is unquestionable. It may be pointed out [Svr], for instance, that the multijet production at the LHC will be dominated by tree level QCD scattering, and at tree level the supersymmetric Yang-Mills theory really does behave like QCD.

In this thesis MHV techniques are reinterpreted in the language of operads. The history behind this part of the thesis is as follows. In a discussion [PFt] on matrix M-theory involving Brannen's idempotents for mass generation [Bra] it was observed that twistor space $\mathbb{CP}^3$ arises naturally in a Jordan algebra setting [Rio] for the unitary ensemble in the matrix model ribbon graph construction of Mulase et al [Wal03]. Both twistors and ribbon graphs may be seen as different aspects of a more categorical formulation of M-theory, in which a Machian principle for topos cohomology partly manifests itself in the dualities of the matrix models. The orthogonal, unitary and symplectic ensembles were shown to exhibit T-duality in [Wal03]. T-duality is taken seriously here as a concrete physical prediction, that information on observables for cosmological scales should be tied to information about observables for the smallest physical scales, but quite independently of its derivation within string theory.

The Machian principle is an abstract holographic principle which does not require an a priori spacetime boundary on which to place fields, since after all these degrees of freedom must also emerge from an underlying measurement geometry. The mathematical boundary principle is recovered most simply by demanding that *all* amplitudes be determined by cohomological pairings. In order to reach the mathematical richness required to encompass the Standard Model it is then necessary to consider general motivic cohomology. Moreover, only in such a universal setting is it possible to find the background independence demanded by this measurement approach. That is not to say that physicists must take on board an enormous new body of mathematics, taking many years to learn, because once suitable computational techniques are extracted their application should be straight forward. Also, as the physics evolves, it should be possible to clarify the cohomology in purely categorical terms. The first part of this thesis looks at how categorical combinatorics, and other category theoretic techniques, offer just such a possibility.

The Veneziano integrals arose in the Regge theory for hadrons. Although the Yang-Mills theory of QCD proved more successful in the 1970s, these integrals lie at the foundations of string theory [Sch75] and still have relevance to hadron phenomenology. In this thesis, this set of integrals is shown to arise naturally from *operad* techniques. If one expects diagram techniques to continue to be useful for physics, then operads are a natural way to imbue diagrams with much more background independent meaning than is contained in Feynman integrals.

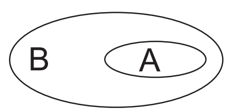
The characterisation of a generation of Standard Model particles by triple ribbon diagrams [BT] further motivates this approach. Categorical ribbon diagrams are dual to the structural diagrams for monoidal categories with duals. Since we know that mass generation breaks the flat space quantum mechanical logic, it is natural to associate the appearance of triple ribbons in [BT] with a richer, as yet poorly understood, categorical structure for which triple ribbons represent not merely duals on objects, but further layers of duality and more general n-alities.

These considerations place unification in the context of a search for a post quantum *topos*. A classical topos [McL92][Gol84][Moe92][Stra] is a category with certain properties that makes it a natural home for the logic of constructive mathematics [EB85], namely intuitionistic logic. For example, the category of sets and functions, **Set**, lives by the rules of Boolean logic. In **Set** the two possible values of truth, true and false, define a two point set and functions into this set characterise subsets. A second example of a topos is any category of sheaves.

The view here is that the higher categorical structure appropriate for quantum logic generalises **Set** [Moe92] not by extending the usual topos axioms to higher dimensions, but rather by considering characterisations of **Set** which already incorporate duals.

Currently there exists no accepted definition of a quantum topos, although the importance of the classical concept is well understood. Ross Street has studied a bicategorical analogue, namely cosmoi [Str74], but the detailed connection between such generalisations and quantum structures has yet to be exposed. There are many motivations for studying an infinite dimensional analogue, such as the infinity toposes of Lurie [Lura], which arose from his definition of elliptic cohomology [Lurb]. This in turn is linked to recent studies in string M-theory [Sat04].

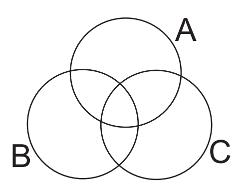
The 19th century mathematical philosopher C. S. Peirce [Pei33] pioneered diagrammatic approaches to logic. In his system of planar graphs the diagram represents the statement that A *implies* B. In other words, the truth of A is



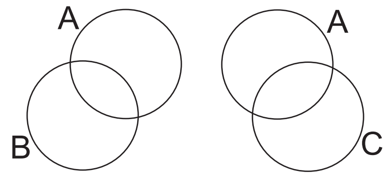
contained in the truth of B, since if A is true then B is definitely true. The region of the plane outside A represents the complement, *not* A. Thus, in classical Boolean logic, two loops about a letter are equivalent to no loops at all, because *not not* A is the same as A.

In the investigation of alternative logics one would also like to have diagrammatic systems which are computationally useful. This problem has been studied by computer scientists such as Cockett and Seely [See03]. But how are we to relate knot and ribbon diagrams to a sensible system of *physical* logic?

For instance, consider distributivity. The statement A *and* $(B$ *or* $C)$ is represented by a subregion of three intersecting circles. Classically, one is free



to slide extra copies of a circle together, forming three circles out of the disjoint union of A *and* B and A *and* C. Is there a different sort of distributivity,



described by moving out of the plane and into diagrams of links or ribbons?

Stringy diagrams are ubiquitous these days and clearly form an integral part of the emerging theory of quantum gravity: in studying the mathematical structure of CFTs [Sch05], in spin foam models [Fre][Per03], in the topological characterisation of QFT particles [BT][Bra] and in the category theoretic idea of stringification [Sch][Pfe]. An analogy between link components and particle number appeared in the theory of topological quantum computing [Lom][Wan02]. One pillar of this enormous body of research is the revolution in our understanding of link invariants brought about by the advent of the Jones polynomial [Por94], and sophisticated new invariants of three and four dimensional manifolds. What is the deeper reason for the power of this knotty mechanics? For a category theorist, the answer lies in the universal nature of categories of knotty diagrams [Str93][Shu94].

In this thesis, braided monoidal categories are used as a basis for studying linear quantum toposes, in a way that aims to recover linear logic without taking it to be fundamental. The issue of premonoidal structures, which is related to the cohomological generation of mass, will only be discussed in chapter 6.

Of course, the application of topos theory to quantum physics has already been studied extensively. With regard to the structure of spacetimes there is the work of Isham [Isha][Ishb][Ishc], Markopoulou [Mar00], Raptis [Rap], Crane and Christensen [Cra05] and many others. From the more computational end, Taylor [Tay] has developed a programme to study Stone duality using intuitionistic ideas.

Quantum logic has also been approached from the perspective of linear logic. Cockett et al [See03] have worked with polycategories rather than ordinary higher dimensional algebra [Bae]. This is a different solution to the problem of introducing two different horizontal compositions as a means of describing the tensor and par of linear logic. In the 1970s, Barr [Bar79] developed the theory of $*$–autonomous categories. This includes compact closed structures, which are the basis of the recent characterisation of quantum computation protocols by Abramsky and Coecke [Coe][Pava]. Yetter's cyclic logic [Yet90] was an early start to a characterisation of non–commutative linear logic [Rue02a].

Rather than begin with topos theory itself, many authors have considered generalising lattice theory to accommodate non–commutative structures such as quantales [Ros03][Res04]. Related in intention to the analysis here are considerations of quantum lattices from a more intuitionistic perspective [Coe02][Cor95]. It will be interesting to compare our simple axioms to the non–commutative logic of [Rue02b], which generalises both linear logic and Yetter's cyclic logic [Yet90]. That is, there are potentially four connectives, a pair of conjunctions and a pair of disjunctions as in

	$\wedge$	$\vee$
commutative	$\wedge_{\mathrm{tr}}$	$\wedge_{\mathrm{fr}}$
non-commutative	$\overline{\wedge}_{\mathrm{tr}}$	$\overline{\wedge}_{\mathrm{fr}}$

for two truth types *true* and *frue*. The non–commutativity of the bottom row arises here from a braiding on the monoidal structure of the category.

At this point, the interpretation of quantum mechanics to keep in mind is a relational one. There is certainly no universal observer in a topos–like logic, in which relationalism is endemic. On the other hand, the ontological nature of classical realities in most relational interpretations must be abandoned. One might well wonder what's left, but this should become clearer.

Chapters 2 and 3 contain background material necessary for appreciating the remainder of the thesis. Chapter 2 introduces the essential concepts of category theory, hopefully in a way that does not assume too much familiarity with them. In chapter 3 we take a motivational look at causality in classical gravity and outline the ideas behind ribbon graph M-theory and the MHV diagram techniques. Chapter 4 gives some background on recent results in universal cohomology and then focuses on the $\mathbb{RP}^1$ localised gluon amplitudes, which may be expressed using cohomological constructions on a

1-operad of moduli spaces. This is followed by a basic starting definition of a quantum topos, which is outlined in chapter 5. The appropriate categories are braided ones. Stringy diagram techniques, although certainly useful, will not be discussed at all here. The logic internal to this notion of quantum topos is considered in chapter 6. Finally, in chapter 7, we pinpoint a few further reasons why one should expect that a refined, higher dimensional, description will be of importance to physics.

In summary, this thesis contains the following new work:

- a new higher categorical view of causality in quantum gravity
- an initial investigation into possible definitions of path integral amplitudes in terms of operad combinatorics
- a study of the relationship between Veneziano n point functions and motivic integrals arising from operad techniques
- an axiomatic definition of a quantum topos
- an analysis of the linear logic associated to these axioms

2 Basic Categories and Toposes

"Now let us say a few words about the important role of paths in metric spaces."

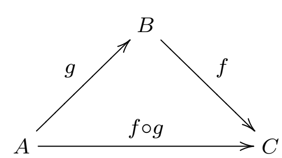
F. W. Lawvere [Law05]

A category expresses relations, or transformations, at a very fundamental level and on an equal footing with the objects on which the maps act. Naturally it is not possible to do this subject justice in one short chapter. The basic aspects of category theory introduced here are those needed in later chapters.

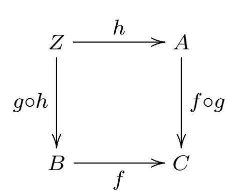
2.1 What is a category?

Whereas a set S has elements, and a map between sets takes elements to elements, a category $\Im$ has both elements, called *objects*, and relationships between elements, called *arrows* or *morphisms*. Every object A is equipped with at least an identity arrow 1_A from A to A which may be identified with the object A itself. Maps between categories, called *functors*, take objects to objects and arrows to arrows in such a way that identities and composition of arrows is preserved.

Arrows $f : B \to C$ and $g : A \to B$ may be composed $f \circ g$ since their ends match appropriately, as in the diagram



It is required that the composition is associative



although this shall no longer hold in the higher dimensional case. The composition symbol $\circ$ will often be omitted. An arrow is *monic*, drawn as

$$B \stackrel{f}{\rightarrowtail} C$$

if for any $g : A \to B$ and $h : A \to B$, $f \circ g = f \circ h$ implies that $g = h$. This is left cancelation. Right cancelation defines a notion of *epic* arrow.

Usually one axiomatises categories by referring to *sets* of arrows between objects, but we deliberately refrain from detailing that particular definition here.

Example 2.1 There is a category **Set** whose objects are sets and whose arrows are functions between sets. In **Set** there is a two element set $\{0, 1\}$. There are also many arrows of the form $f : S \to \{0, 1\}$ for a set S. Such arrows may be thought of as the selection of a subset of S, namely those elements that are mapped to 1. A one element set, $\{*\}$, has precisely one arrow into it from any other set, making it an example of a *terminal object* **1** in **Set**.

Example 2.2 Any poset is a category with elements as objects and an arrow $X \to Y$ whenever $X \leq Y$.

A *faithful* functor $F : \Im \to \mathcal{C}$ is one for which, given a pair $f, g : A \to B$ of arrows in $\Im$, $F(f) = F(g)$ implies that $f = g$.

Functors are contravariant if they actually act on $\Im^{\mathrm{op}}$ rather than $\Im$, where $\Im^{\mathrm{op}}$ is the same category as $\Im$ but with all the arrows symbolically reversed. The reversal of particular diagrams defines *dual* concepts. For example, epic arrows are dual to monic arrows.

Contravariant functors from a (small) category $\Im$ into **Set** are known as *presheaves*, the total collection of which provide a preliminary example of a topos. When $\Im$ comes equipped with a topology one restricts to a subcategory of *sheaves*. In particular, for a topological space X let $O(X)$ be the category whose objects are the open sets and whose arrows are the inclusion homeomorphisms. Observe that $O(X)$ has a terminal object, namely the set X itself. The empty set is initial. A presheaf is a functor from $O(X)^{\mathrm{op}}$ into **Set**.

There is no reason to restrict one's attention to functors (which are 1-dimensional arrows) between categories. In analogy with the way that the step from sets to categories took us from dimension 0 to dimension 1 we find that there are 2-dimensional maps between functors. These are *natural transformations* [Mac00]

$$\Im_1 \underset{G}{\overset{F}{\Downarrow\eta}} \Im_2$$

between functors F and G. Such 2-arrows are specified by a family of commuting squares in the target category $\Im_2$, namely

$$\begin{CD} F(A) @>{F(g)}>> F(B) \\ @V{\eta_A}VV @VV{\eta_B}V \\ G(A) @>>{G(g)}> G(B) \end{CD}$$

with an arrow η_A associated to each object A in $\Im_1$.

Example 2.3 Determinants [Mac00] are really a natural transformation between the two functors $GL_n : \mathbf{Rng} \to \mathbf{Grp}$ and $(\_)^* : \mathbf{Rng} \to \mathbf{Grp}$ from the category of (commutative) rings to the category of groups. The first functor assigns the obvious matrix group to a ring, and the second the group of units K^* to a ring K. This fact hints at the need to consider even elementary constructs from linear algebra in a setting where the base number field is not fixed a priori.

Example 2.4 Let $\mathbf{FinVect}_K$ be the category of finite dimensional vector spaces over the field K. There is a category $\mathbf{Mat}_K$ whose objects are the ordinals $n \in \mathbb{N}$ and whose arrows $m \to n$ are the $m \times n$ matrices over K. A functor $\mathbf{FinVect}_K \to \mathbf{Mat}_K$ picks out the dimension of the vector space V and the linear maps of $\mathrm{End}(V)$ are expressed in terms of a choice of basis. Given two such functors B_1 and B_2 a natural transformation $\eta : B_1 \Rightarrow B_2$ is a change of basis η_V for each vector space, as a similarity transformation between matrices.

The intended interpretation of pieces of categories is that they are geometric entities. In a category there is no equality between objects, but we consider objects isomorphic if there exist two arrows f and g as in

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

such that $f \circ g = 1_A$ and $g \circ f = 1_B$. Equivalence in dimension 2 is naturally weaker again. The important relationship between two categories $\Im_1$ and $\Im_2$ and two functors $F : \Im_1 \to \Im_2$ and $G : \Im_2 \to \Im_1$ is that of an *adjunction*, where one only requires that FG and GF are like the identity functor up to natural isomorphisms. That is, an adjunction $F \dashv G$ has data $(F, G, \eta, \varepsilon)$ where $\eta : GF \Rightarrow 1_{\Im_1}$ and $\varepsilon : FG \Rightarrow 1_{\Im_2}$ are natural transformations.

Example 2.5 **Vect** is the category of all vector spaces over a field K. Let $A : \mathbf{Set} \to \mathbf{Vect}$ be the functor that assigns to a set S the formal linear combinations of elements of S. There is a functor $B : \mathbf{Vect} \to \mathbf{Set}$ that forgets the vector space structure of V and treats the vectors of V simply as a set. In the adjunction $A \dashv B$, $\eta : BA \Rightarrow 1_{\mathbf{Set}}$ is specified by arrows η_S that project $S \times K$ onto S. The natural transformation $\varepsilon : AB \Rightarrow 1_{\mathbf{Vect}}$ identifies the highly degenerate representations of vectors as linear combinations of all vectors in V with the vector itself. Note that the functor A takes a Cartesian product $X \times Y$ of sets to the tensor product $A(X) \otimes A(Y)$ in **Vect**, because a basis for the tensor product space is given by ordered pairs of elements from X and Y.

Example 2.6 Let **RMod** be the category of left R-modules for a ring R, and **RModS** the category of RS-bimodules. The forgetful functor **RModS** $\to$ **RMod** has as left adjoint the functor sending V to $V \otimes S$ [Mac00].

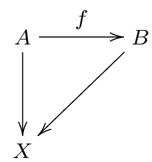
Consider in a category $\Im$ an object X together with a family of arrows $\{f_i : X_i \to X\}_{i \in I}$ indexed by I. A family of two such arrows is a *cospan*. A *sieve* is a family where, given any $x : Y \to X_i$, $f_i \cdot x$ is also in the family. Given a faithful functor $\Im \to \mathcal{C}$ a *final sink* [Stra] is such a family where, given any $x : F(X) \to F(Y)$, x is in $\Im$ whenever $F(f_i) \cdot x$ is.

Example 2.7 A terminal arrow $A \to 1$ is a simple final sink.

Example 2.8 The dual notion to that of final sink is that of an *initial source*. Monics in **Vect** are single arrow initial sources.

Example 2.9 An *Alexandroff space* X (not to be confused with an Alexandrov space) in **Top** is a finitely generated space. That is, the family $\{X_i \rightarrowtail X\}$ of finite subspaces of X forms of a final sink. For an Alexandroff space, every point has a smallest open neighbourhood; also, the associated preorder $(X, \leq)$, defined by $x \leq y$ if and only if x is in the closure of y, completely determines the topology. That is, the category of Alexandroff spaces is equivalent to the category **Pre** of preorders and monotone maps.

A *slice category* $(\Im, X)$ based at an object X has objects the arrows into X and arrows f the commutative triangles



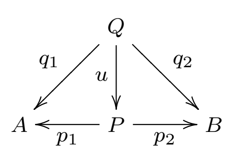
Observe that the identity 1_X acts as a terminal for $(\Im, X)$. This is actually a special case of a more general notion of comma category (F, G) for functors $F : \mathcal{C} \to \Im$ and $G : \mathcal{D} \to \Im$ of target $\Im$ [Mac00].

2.2 Pullback Lemmas and Sheaves

Given objects A and B in $\Im$ there may exist a span diagram [McL92]

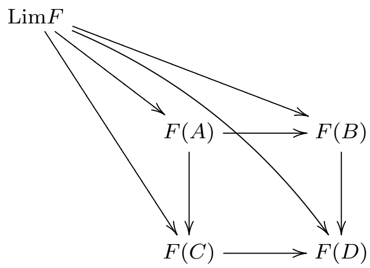
$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

The (Cartesian) product $P = A \times B$ is such a span with the *universal property* [Mac00] in that, given any other span Q, there exists a unique arrow $u : Q \to P$ such that

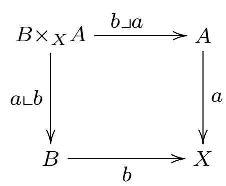


commutes. The Cartesian product of objects allows us to define functors of the form $\_ \times B : \Im \to \Im$. There is a notion of *Cartesian closed* category where right adjoints to all these functors exist.

More generally, the commuting figure



is a cone in the target category of the functor F. A cone vertex that is universal with respect to any other cone is called Lim$F$. This is the way category theorists like to view limits. A universal cone over a cospan $B \to X \leftarrow A$ gives the *pullback* square



An *equaliser* is an arrow e in a diagram

$$E \xrightarrow{\ e\ } A \overset{x}{\underset{y}{\rightrightarrows}} B$$

such that $xe = ye$ and, given any arrow $f : Q \to A$ with $xf = yf$, there exists a unique arrow $Q \to E$ making the triangle commute.

Example 2.10 Given a diffeomorphism $f : M \to N$ between manifolds, the pullback $f^* : \Omega^0(N) \to \Omega^0(M)$ of functions on N characterises the functor Ω of differential forms.

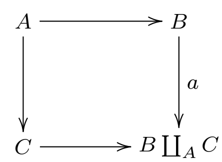
Definition 2.11 We consider a category $\Im$ with a product $\wedge$, which defines a partial order as for intersections, such that each collection of objects X_i has a supremum X, or least upper bound, in $\Im$. A *sheaf* is a presheaf F on $\Im$ satisfying the condition that

$$F(X) \longrightarrow \prod_i F(X_i) \overset{i_1}{\underset{i_2}{\rightrightarrows}} \prod_{ij} F(X_i \wedge X_j)$$

is an equaliser, where the i arrows come from inclusions $X_i \wedge X_j \to X_i$ in $\Im$.

Example 2.12 The open sets of the celestial sphere $O(S^2)$ with $\wedge$ the intersection of sets has on it a sheaf of germs of holomorphic functions, usually defined as functions on open sets about a point $x \in S^2$ with the equivalence relation $f \sim g$ if there exists a neighbourhood U of x such that $f = g$ on U [Jnr90]. The equaliser condition says that for the restriction arrows $r_{i,ij} : F(U_i) \to F(U_i \wedge U_j)$, if $r_{i,ij}(f_{U_i}) = r_{j,ij}(f_{U_j})$ for all i, j then there exists a global f. Moreover, if two elements of $F(U)$ are the same when restricted to any U_i in U then it follows that they are identical.

The dual notion to pullback is that of a *pushout*, as in

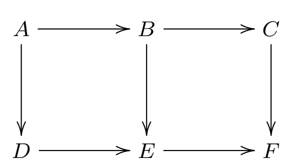


Pullbacks and pushouts abound in the categorical structures that we will be considering. The following elementary results, where obvious proofs are omitted, will be used frequently in later chapters often without statement.

Lemma 2.13. *The pullback $a \llcorner b$ of a monic arrow $a : A \rightarrowtail X$ along an arrow $b : B \to X$ is monic.*

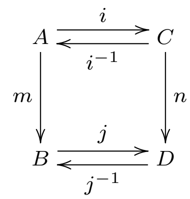
Proof. Assume that $(a \llcorner b)x = (a \llcorner b)y$. Then $a(b \lrcorner a)x = b(a \llcorner b)x = b(a \llcorner b)y = a(b \lrcorner a)y$, so that $(b \lrcorner a)x = (b \lrcorner a)y$ since a is monic. By the uniqueness property of the pullback, this implies that $x = y$. $\square$

Lemma 2.14. *In the diagram*



if both the right hand square and outside rectangle are pullbacks then so is the left hand square.

Lemma 2.15. *A commuting square*

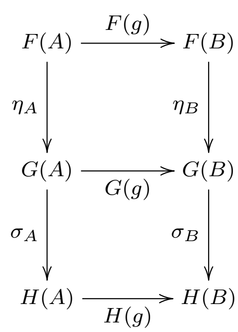


with invertible arrows as shown, is always a pullback.

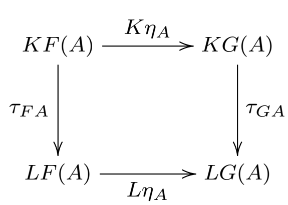
Proof. Take arrows $\beta : Q \to C$ and $\alpha : Q \to B$ such that $\alpha = j^{-1} \cdot n \cdot \beta$. Then $\alpha = j^{-1} \cdot n \cdot i \cdot i^{-1} \cdot \beta = m(i^{-1} \cdot \beta)$. In other words, the arrow $i^{-1} \cdot \beta$ is the only arrow $Q \to A$ that makes the diagram commute. $\square$

2.3 Two Dimensional Structures

Implicitly we have been discussing a category **Cat** with categories as objects and 1-arrows the functors between them. Cartesian product for categories makes sense, with $(f, g) \in \Im_1 \times \Im_2$ defined by ordered pairs as in **Set**. One may naturally include in this category the natural transformations between functors. The natural transformation squares for $\eta : F \Rightarrow G$, $\sigma : G \Rightarrow H$ and $\tau : K \Rightarrow L$ may be composed, both vertically



and horizontally



Thus **Cat** is an example of a 2-category, an inherently two dimensional structure. In a 2-category, all arrows between two objects A and B, denoted

$\Im(A,B)$, form a category rather than a set because there are two levels of arrows on top of the identity arrows, and composition has been suitably defined.

Example 2.16 We would like to consider a 2-category n-**Cob** which has as 2-arrows n-dimensional manifolds with corners, with boundary component 1-arrows being $(n-1)$-dimensional manifolds with $(n-2)$-dimensional boundaries.

Consider a 2-category $\Im$ with only one object $*$. Now think of the 1-arrows as objects in the category $\Im(*,*)$. The identity arrow 1_* becomes the *unit* object I. The 2-categorical composition of objects in $\Im(*,*)$ is thought of as a tensor product structure and the 2-arrows are then the 1-arrows with composition the vertical composition of the 2-category.

Definition 2.17 [Kas95] A *monoidal category* $\Im$ consists of

1. a functor $\otimes : \Im \times \Im \to \Im$
2. an associativity natural isomorphism $\psi : \otimes(\otimes \times 1_\Im) \to \otimes(1_\Im \times \otimes)$
3. a left unit natural isomorphism $\lambda : \otimes(T \times 1_\Im) \to 1_\Im$ with respect to a unit object T such that

$$\begin{array}{ccc} T\otimes A & \xrightarrow{\lambda_A} & A \\ {\scriptstyle 1_T\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ T\otimes B & \xrightarrow[\lambda_B]{} & B \end{array}$$

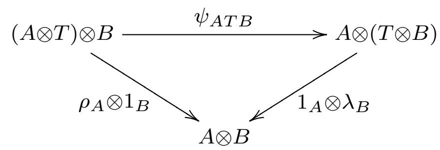
   for any arrow f

4. a right unit natural isomorphism $\rho : \otimes(1_\Im \times T) \to 1_\Im$ with a similar condition on arrows f with respect to ρ_A and ρ_B

such that the associativity arrows $\psi_{ABC} : (A \otimes B) \otimes C \Rightarrow (A \otimes B) \otimes C$ obey the Mac Lane pentagon [Mac00]

$$\begin{array}{ccccc} (A\otimes B)\otimes(C\otimes D) & \longrightarrow & ((A\otimes B)\otimes C)\otimes D & & \\ & & & \nwarrow & \\ \uparrow & & & & (A\otimes(B\otimes C))\otimes D \\ & & & \nearrow & \\ A\otimes(B\otimes(C\otimes D)) & \longrightarrow & A\otimes((B\otimes C)\otimes D) & & \end{array} \tag{1}$$

and the triangle rule

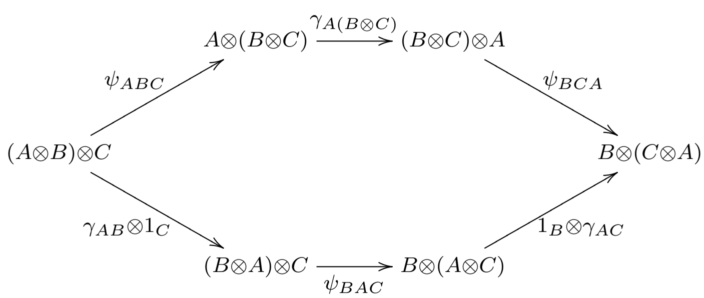


(2)

Example 2.18 **Set** with the usual Cartesian product as a tensor product.

Example 2.19 The category **Vect** with tensor product of vector spaces.

A *braided monoidal* category has the extra structure of a natural isomorphism $\gamma : \otimes \to \otimes{\cdot}\tau$ where τ flips the entries of the ordered pair. In other words, for each pair of objects A and B there is an isomorphism $\gamma_{AB} : A \otimes B \to B \otimes A$. These arrows must satisfy the hexagon



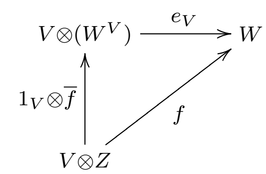
(3)

and another similar hexagon diagram. Finally, a *symmetric monoidal* category is a braided monoidal category for which $\gamma_{AB} \cdot \gamma_{BA} = 1_{B \otimes A}$ for all objects A and B. This ad hoc sounding definition will be somewhat clarified when we look at such structures in their higher dimensional guise later on.

Example 2.20 The category **Vect** equipped with the symmetric operation of tensor product of vector spaces.

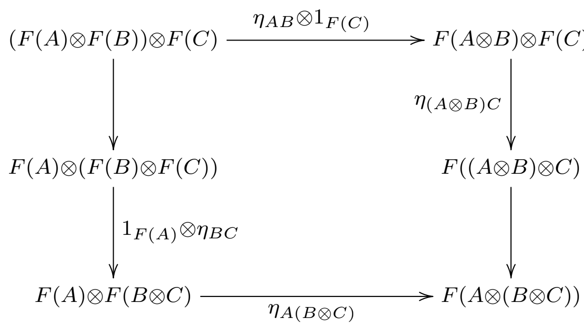
Example 2.21 The non-negative real numbers $\mathbb{R}_0^+ \cup \infty$ form a symmetric monoidal category with $\otimes$ given by addition and arrows given by the reverse ordering $\geq$ [Law02]. Truncated subtraction provides adjoints to the functors $X \otimes \_$, making this category *monoidal closed*.

Monoidal (right) closure is expressed by the existence of objects W^V and arrows $\overline{f} : Z \to W^V$ for any given $f : V \otimes Z \to W$ such that the triangle



commutes.

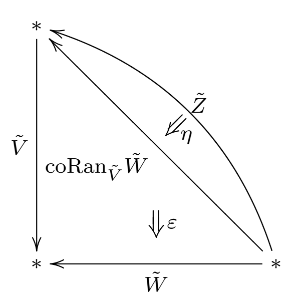
A *monoidal functor* between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a pair (F, η) where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $\eta_{AB} : F(A) \otimes F(B) \Rightarrow F(A \otimes B)$ a natural transformation satisfying $\eta_{AI} = \eta_{IA} = 1_{F(A)}$ up to the isomorphisms λ and ρ, and such that the diagram



(4)

commutes.

What is the 2-dimensional analogue of the monoidal closure diagram? Since objects in a monoidal category are really 1-arrows there should be four arrows in the diagram. Similarly, there should be three faces on the closed surface. By Euler's theorem this means that the number of vertices will be $v = 2+4-3 = 3$. The only possible conclusion, up to duality, is a diagram of the form



known as a *coKan extension*, or right lifting [Kel05].

2.4 Internalisation

In this new higher dimensional world, we have seen that *sets* are really nothing more than categories with only identity arrows. But the vast majority of mathematics used in physics is clearly set theoretic. That's all very well of course, so long as we now acknowledge the context of this mathematics, namely the category **Set**. From now on a *group*, for instance, will not be merely a stand alone set equipped with extra structure, but rather a certain collection of diagrams of arrows in **Set**. Arrows, in other words, are not permitted to hold mathematical meaning independently of a precise semantic context.

To begin with, note that the objects of any kind of category may be replaced by an arrow in **Cat**. The trivial category **1** in **Cat** is the category with only one object and one arrow, the identity arrow. Any functor from **1** into a category $\Im$ picks out an object and its identity arrow. Similarly, in any category with a terminal object **1**, an arrow from **1** to an object X represents an element of X.

Example 2.22 In the category **Top** of topological spaces, a point in a space X is represented by an arrow $\mathbf{1} \to X$ from the one point space.

Example 2.23 In **Set** the evaluation of a function $f : B \to C$ at x looks like a composition

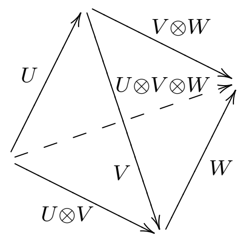
$$\mathbf{1} \xrightarrow{x} B \xrightarrow{f} C$$

Categories themselves, at least small ones, may be viewed as internal structures in **Set**, as we will see below.

Define a category $\mathbf{\Delta}$ [Mac00] with objects the finite ordinals $n \in \{0, 1, 2, \cdots\}$ and arrows as in the example of the truncated subcategory representing the ordinal **4**,

$$0 \xrightarrow{\delta_0^0} 1 \underset{\delta_1^1}{\overset{\delta_0^1}{\rightrightarrows}} 2 \underset{\delta_2^2}{\overset{\delta_0^2}{\rightrightarrows\!\!\!\to}} 3 \tag{5}$$

that is, the order preserving functions from i to j in n. Such a diagram is better visualised as the six edged 3-simplex, an example of which



expresses associativity. Observe that such simplices are equipped with oriented edges and faces [Str87]. The labels on the tetrahedron indicate that it is really a piece of a monoidal category, with 1-dimensional objects. In general, a *simplicial object* of $\Im$ is a contravariant functor $\mathbf{\Delta}^{\mathrm{op}} \to \Im$.

This is but one instance of the more general idea: defining structure in terms of diagrams internal to the category of interest. Let us return to a simple example. A *group object* in **Set** is defined by diagrams involving the operations multiplication $m : G \times G \to G$, unit $e : \mathbf{1} \to G$ and inverse $i : G \to G$, namely associativity

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{1_G\times m} & G\times G \\ {\scriptstyle m\times 1_G}\downarrow & & \downarrow{\scriptstyle m} \\ G\times G & \xrightarrow[m]{} & G \end{array} \tag{6}$$

inverse laws

$$\begin{array}{ccccc} G & \xrightarrow{\Delta} & G\times G & \xrightarrow{i\times 1_G} & G\times G \\ \downarrow & & & & \downarrow{\scriptstyle m} \\ \mathbf{1} & & \xrightarrow[e]{} & & G \end{array} \tag{7}$$

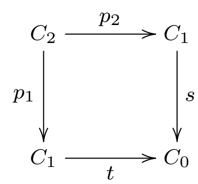
and unit law

$$\begin{array}{ccccc} \mathbf{1}\times G & \xrightarrow{e\times 1_G} & G\times G & \xleftarrow{1_G\times e} & G\times\mathbf{1} \\ & \searrow{\scriptstyle \simeq} & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \simeq} & \\ & & G & & \end{array} \tag{8}$$

An object G satisfying these laws is an ordinary group. The lesson of internalisation is that these diagrams make sense *in any category* with finite products.

Observe that a group is really just a one object category with invertible arrows. The category of groups is therefore a collection of functors between one object categories. A *groupoid* is any category in which all arrows are invertible. It is known that higher dimensional analogues of groupoids, namely weak n-groupoids, are sufficient to model homotopy types.

For two objects C_0 and C_1, and source and target maps $s : C_1 \to C_0$ and $t : C_1 \to C_0$ on the arrow object C_1, we can define a (strict) *category object* by the diagram



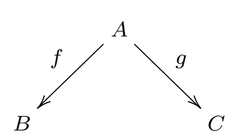
(9)

where $C_2 = \{(f, g) \in C_1 \times C_1 : tf = sg\}$ is a pullback and the p_i are projections. Laws for associativity and identities are similar to those above.

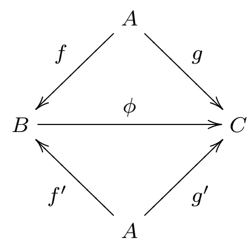
Geometrically, these laws are closed surfaces; circles in the case of the group laws. They therefore belong in an object of one higher dimension than the arrows comprising them. That is why for category objects there is a 2-dimensional object C_2. One can then think of an internal category as a truncated simplicial object on four objects C_0, C_1, C_2, C_3, with the single object $*$ as C_0 and the laws involving C_3.

There is another way of defining category objects in a category $\Im$ which is more pertinent to the physical constructions that appear later. This requires the more general 2-categorical structure of *bicategory*, which is defined in more detail in the appendix.

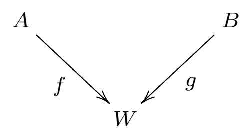
To begin with, the alternative approach requires the bicategory $\mathbf{Spn}(\Im)$ of spans in a category $\Im$ which has pullbacks. The objects of $\mathbf{Spn}(\Im)$ are the objects of $\Im$. A 1-arrow $b : B \to C$ is a diagram of the form



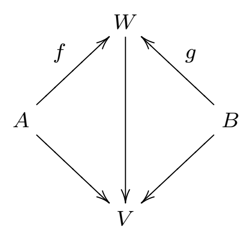
and a 2-arrow $\phi : b \Rightarrow c$ for $b, c : B \to C$ is a commuting diamond



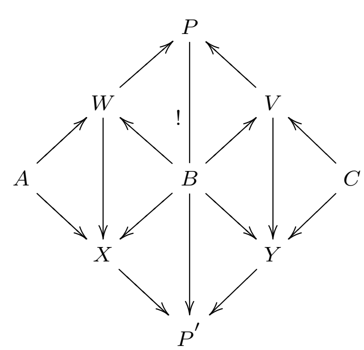
Similarly, there is a dual bicategory $\mathbf{Cosp}(\Im)$ of cospans in a category $\Im$ with pushouts. The 1-arrows are cospans



and the 2-arrows are diamonds



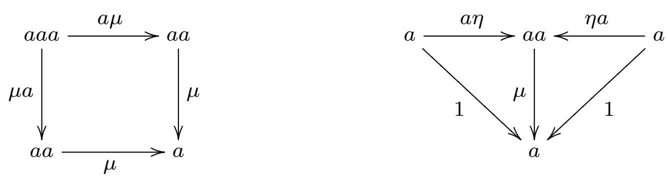
Horizontal composition of 1-arrows in $\mathbf{Cosp}(\Im)$ is given by the pushout P of the internal arrows, and the defining 1-arrow of the horizontal composition of 2-arrows is given by the unique arrow from the pushout P into P' in the diagram



(10)

Vertical composition is by ordinary composition of the interior 1-arrows. The interchange law (see appendix) follows straightforwardly from the uniqueness of the composition of two such pushout arrows. Composition in **Spn**($\Im$) is similar but in terms of pullbacks rather than pushouts.

An internal category in $\Im$ is then a *monad* in **Spn**($\Im$). A monad [Str72] in any 2-category is an arrow $a : X \to X$ together with 2-arrows $\eta : 1_X \Rightarrow a$ and $\mu : aa \Rightarrow a$ satisfying

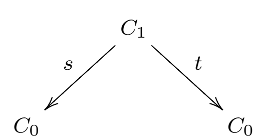


(11)

Observe the similarity with the associativity and unit laws above.

Example 2.24 The physical concept of a *causal set* underlies one approach to quantum gravity. Causality is represented by arrows, making causal sets into categories. Let **Chron** be the category in **Cat** of causal sets and future–limit preserving monotone functions [Har00], where a point x is a future–limit if there exists a chain c of points such that $I^-(x) = I^-(c)$. The addition of a future boundary is a functor $+ :$ **Chron** $\to$ **Chron**. Clearly, this functor satisfies $+^2 = +$. There is a natural transformation $1 \Rightarrow +$, making the functor a monad.

In **Spn**($\Im$), $a : B \to C$ is a span in $\Im$ and η and μ are specified by $\Im$ arrows $B \to C$. Choosing a span suggestively of the form



the monad laws do indeed axiomatise associativity of composition of elements of C_1.

A distributive law [Str72] is an arrow describing the commutativity of monads, as in $ab \Rightarrow ba$ for monads a and b.

Example 2.25 For the category **Set** let a be the free monoid monad, which describes multiplication, and b the abelian group monad, which describes addition. Then the distributivity of addition and multiplication is a natural transformation $ab \Rightarrow ba$.

2.5 What is a topos?

It is now time to introduce the notion of a *topos* [Joh77][Gol84][McL92][Wyl91], due to Lawvere and Tierney, going back to ideas of Grothendieck. This is the classical model for the axioms that will be studied in chapters 5 and 6.

Definition 2.26 A *topos* is a Cartesian closed category with pullbacks, a terminal object $\mathbf{1}$, a *subobject classifier* object $\mathbf{\Omega}$ and a special arrow, called true, such that for any monic arrow from A to B there exists a unique χ creating the pullback

$$\begin{array}{ccc} A & \rightarrowtail & B \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi} \\ \mathbf{1} & \xrightarrow[\text{true}]{} & \mathbf{\Omega} \end{array} \tag{12}$$

Example 2.27 The category **Set** of sets and functions. The subobject classifier is the two point set $\{0, 1\}$. The arrow true selects the element 1 from this set. For any monic representing a subset, there is a characteristic function χ into $\{0, 1\}$ which sends precisely the elements of the subset to 1.

Example 2.28 The category of all sheaves on the celestial sphere (see chapter 3) including the germs of holomorphic functions on the sphere.

Example 2.29 The category of functors from $\mathcal{C}$ into **Set** for any small category $\mathcal{C}$. When $\mathcal{C}$ is a groupoid the logic of this topos is Boolean [Moe92].

The Lawvere–Tierney Theorem [Joh77][Gol84] states that for any object X of a topos $\mathcal{C}$, the comma category $(\mathcal{C}, X)$ is also a topos.

Geometric morphisms [Moe92] between toposes are pairs of functors, $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$, which are adjoint and such that G is finite limit preserving. Such morphisms are the natural arrows in a 2-category of all toposes.

A *topology* on a topos may be expressed most easily as a choice of arrow $j : \mathbf{\Omega} \to \mathbf{\Omega}$ and commuting diagrams

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\mathrm{tr}} & \Omega \\ & {\scriptstyle \mathrm{true}} \searrow & \downarrow {\scriptstyle j} \\ & & \Omega \end{array} \qquad \begin{array}{ccc} \Omega & \xrightarrow{j} & \Omega \\ & {\scriptstyle j} \searrow & \downarrow {\scriptstyle j} \\ & & \Omega \end{array} \qquad \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ {\scriptstyle j \times j} \downarrow & & \downarrow {\scriptstyle j} \\ \Omega \times \Omega & \xrightarrow[\wedge]{} & \Omega \end{array} \tag{13}$$

Example 2.30 The logical operation of NOT in a topos is an arrow $\Omega \to \Omega$ and the composition of two copies of this negation arrow is a topology in this axiomatic sense [Moe92].

The representation of a topology by an arrow mirrors the universal localisation functor from a category of presheaves to the associated sheaves. Given a subcollection S of arrows in $\Im$ one defines the localisation category $S^{-1}\Im$ by sending all arrows in S formally to isomorphisms under a functor $\Im \to S^{-1}\Im$ which has the universal property. Equivariant localisation was used by Witten [Wit92] to evaluate a two dimensional Yang-Mills path integral exactly. Physically, one would like to understand a four dimensional non-commutative analogue to this localisation process.

Remark 2.31 Categories in Representation Theory are derived from the Boolean toposes of functors from a group G into **Set**, by taking the vector space objects of the topos [Moe92]. This might suggest that an appropriate context in which to study quantum mechanics is the 2-category of all Boolean toposes. However, here we do not take seriously the idea that classical symmetry, or even its quantum group counterpart, is fundamental to a description of physical states. The usual linear categories of vector spaces or Hilbert spaces are no more than convenient *models* of physical logic, in the sense of Lawvere.

2.6 The Combinatorics of Operads

Batanin [Bata][Batb] has recently worked out the coherence law polytopes in a weak n-category setting, for all dimensions. The combinatorics of operads provide the framework for our study of gluon amplitudes in twistor inspired M-theory [Wit04a].

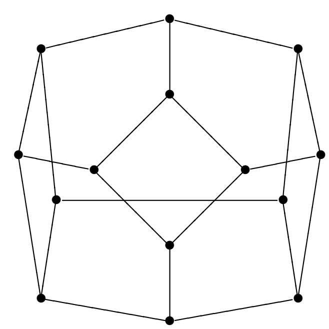
An *operad* is a monoid object in a monoidal category. More precisely, a 1-operad is a collection of spaces $\{K_d\}$, for d an ordinal, with rules for composition of the form

$$K_m \otimes K_{d_1} \otimes K_{d_2} \otimes \cdots \otimes K_{d_{m-1}} \longrightarrow K_n$$

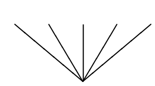
where $n = \sum d_i$, given any order preserving map $\sigma : n \to m$ such that $\sigma^{-1}(i) \simeq d_i$. These rules satisfy suitable associativity and unit conditions. It is easiest to picture the composition maps as a two level tree with d_i leaves attached to a leaf of a tree with m leaves. That is, ordinals are represented by 1-level trees and the gluing on the two level tree is described by the map σ.

One example is the operad of moduli of d-punctured spheres, which appears in conformal field theory. Another is the sequence of d-dimensional Stasheff associahedra characterising 1-fold loop spaces. The associahedron is the polytope whose vertices are labelled by all possible bracketings of binary rooted trees with $(d+2)$ leaves.

Observe that in two dimensions this associahedron is the Mac Lane pentagon. The 3-dimensional polytope, without labels, looks like



By contracting one edge so that one vertex becomes ternary, one also labels edges by trees. Thus the whole polytope is labelled by the simple 1-level tree



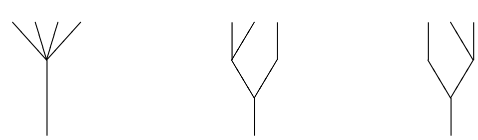
In order to work with integrals over such polytopes we need a realisation of them in $\mathbb{R}^n$. This was worked out in all dimensions by Loday in [Lod04]. The example of the three dimensional polytope illustrates the general method. Let the permutations S_4 on four letters determine integral coordinates for points in $\mathbb{R}^4$. For instance, the points $(1,2,3,4)$ and $(2,4,3,1)$ are permutations. These 24 points all lie on the hyperplane

$$x_1 + x_2 + x_3 + x_4 = 10$$

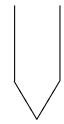
The convex hull of these points is the Stasheff polytope, and it is dual to a cell decomposition of $\mathbb{R}^3$ which naturally appears as a local model for metric ribbon graph moduli [Pen]. A metric ribbon graph has edges labelled with positive real numbers. There is a theorem stating that the Cartesian product of the orbifold $\mathcal{M}_{g,n}$ with a cone $\mathbb{R}^n_+$ is isomorphic to a moduli of metric ribbon graphs associated to g and n. This will be important to the formulation of M-theory.

So in the 1-operad setting, an ordinal d may be represented by such a one level tree. Higher level trees label laws for more general n-operads. The higher operads of Batanin [Bata] produce, for example, the coherence laws that underlie fusion and braiding laws for anyons. An anyon is considered a point particle in a plane. Since Batanin works with real spaces $\mathbb{R}^n$ for n-operads it is natural to begin with the usual planar configurations in terms of a 2-operad. This is in contrast to the usual use of the 1-operad of moduli of punctured spheres, but is more natural since the pentagon and hexagon rules are given by two level trees, which index 2-operads.

Coherence laws for 3-operads are all polytopes in three dimensions. For 2-operads sitting inside 3-operads [Bata] the only 6 edged trees corresponding to planar polytopes are



giving the pentagon associahedron and two hexagons respectively. Restricting attention to planar polytope laws, but allowing the categorical dimension to increase by one, there is precisely one extra 3-operad law coming from the *symmetry tree*



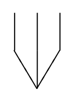
This law provides the symmetry rule for symmetric tensor categories, which appear at the stable level on the list of 1-categorical rules [Dol98]. It describes a circle made of two edges.

Example 2.32 Consider the loop braid group describing exotic statistics for closed string loops moving in $\mathbb{R}^3$, as studied in [Cra]. The loop braid group is described by both braid group generators σ_i and symmetry generators s_i which satisfy $s_i^2 = 1$. The former describe configurations of loops moving through each other in $\mathbb{R}^3$, which is a suspension of the usual point particle anyons in $\mathbb{R}^2$.

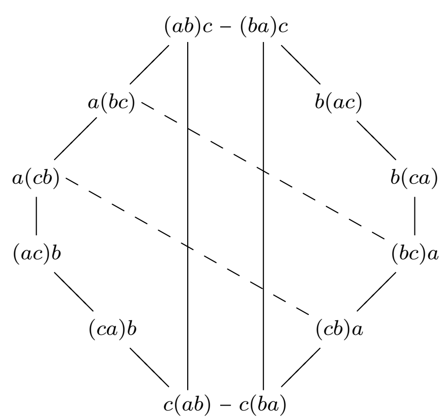
The relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{14}$$

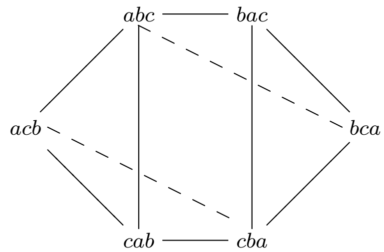
come from another six edged two level tree with only three inputs, namely the Breen tree



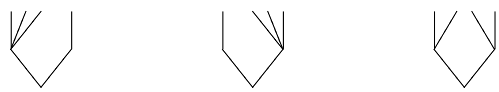
which results in a 3-dimensional polytope made of four hexagons and two squares.



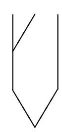
In fact, the Batanin compactification expands the twelve outer edges into squares. The Yang-Baxter rules follow from collapsing the associativity edges, which are assumed to be identities, and associating alternately to the braiding edges the two positive B_3 generators σ_1 and σ_2. This results in a hexagon with diagonals



But the Breen polytope and the suspended Stasheff associahedron are by no means the only coherence rules for 2-operads in three dimensions. Further laws on four inputs, known as *resultoassociahedra* [Batb], appear as 3-dimensional polytopes. These correspond to the trees



and they are added to the 3-operad rules via the addition of a root edge. That brings us up to five coherence rules for 3-operads. We have forgotten to consider the two seven edged trees of the form



which each describe a punctured cylinder (see below) and give a hexagon law for the symmetry generators s_i. This completes the list of 3-dimensional laws for 3-operads. Note that stability for bicategories appears at the 4-operad level with the addition of only one extra rule given by a suspended symmetry diagram, namely a sphere composed of two discs.

Are there any other three input trees for 3-operads? There are two more possible such tree types, namely

which describe polytopes in dimensions four and five respectively.

Higher operad polytopes are important for understanding higher cohomology, where coefficients lie in a general category and we require a generalisation of the concept of simplicial object as a labelling of triangular simplices. Strict n-categories are described by Street's oriented simplices [Str87] but weak n-categories require more shapes, and operad polytopes are needed to describe these, just as they define weak n-categories themselves.

An *algebra* for a 1-operad is a map, preserving the operad structure, from the operad into a vector space operad given as follows. Let V be a given vector space over a field. For the n-fold tensor product $V^{\otimes n}$ there is the space of endomorphisms $E_n \equiv \mathrm{End}(V^{\otimes n}, V)$. This sequence E_n forms an operad under the composition of maps, and provides a product on V.

3 MHV Amplitudes and Ribbon Graphs

"Now would I have a book where I might see all characters of planets of the heavens, that I might know their motions and dispositions."

C. Marlowe *Doctor Faustus*, 1624

One is quite used to the notion of indistinguishibility for fundamental particles in the Standard Model. Some time ago, Bekenstein [Bek74] suggested that quantum gravity should similarly be constructed out of a notion of fundamental state for spacetime degrees of freedom. But it is difficult to remove oneself from the prejudices of the classical picture for spacetime, be it continuous or discrete.

Consider the observation of the CMBR. One observes photons locally. We infer that the photons come from far away, or rather from long ago. What does this mean? Let us assume for the moment that the temperature of the CMBR defines a notion of universal time. The hotter its local measurement, the earlier in the history of the universe we believe ourselves to be. Near an active black hole, for instance, observers may be forced to conclude they live in the early universe. Similarly, any high energy environment may be early in universal time, in the sense that it is high energy photons that are observed. One might prefer to use some measure of entropy as an alternative time variable. Now it should be clear that it is not the usual notion of time being referred to here, since its measure may vary for different classes of observer even here on Earth. It suggests rather a concept of quantum boundary to spacetime observables, since already there is a T duality between early and late times at horizons.

In a measurement based theory, there is no concept of space or time without matter. How can QFT accommodate cosmological notions of time? Feynman techniques for QFT yield scattering amplitudes to extraordinary experimental precision, but rely on renormalisation procedures involving ill defined quantities. These infinities arise as a result of the continuum background of the theory. From a quantum gravitational perspective one would like a completely different procedure for producing the same results, which replaces the renormalisation procedure by gravitational degrees of freedom in such a way that infinities never arise. Given the success of Connes' program [Con][Kre00] in describing the Hopf algebra structure of renormalisation, one seeks alternatives with the potential of recovering such structures in the flat space domain.

In general, the growing significance of diagram techniques in mathematics begs the question of whether or not the traditional path integrals, based on Lagrangians, should be replaced by more direct diagram invariants. Physically, however, this would require replacing the group theoretic basis of QFT entirely, and is therefore a highly non-trivial issue. Nonetheless, since *some* theoretical stance is essential, this is the point of view adopted here.

The key issue not dealt with by QFT is that of mass generation. The Higgs boson remains unobserved and allows massive states only through parameterisation. In the measurement philosophy of QFT, originally promoted by Schwinger, the concept of a vacuum as a seething pool of virtual particles should be replaced by a concept of genuine nothingness. Virtual particles are potentially as real as any other, in as far as particles earn existence only through measurement.

On the other hand, the advanced techniques of Hopf algebra renormalisation theory [Kre00] offer important insights into actual physical tools. But Hopf algebras in general are very categorical gadgets, since they provide the standard examples of braided monoidal representation categories. Thus one expects Hopf algebras to arise but second to the categorical logic dictated by the underlying causality. For this reason, the new MHV twistor techniques are considered as far as possible from a gravitational point of view.

We begin by discussing a certain approach to mass generation from the 1980s within Penrose's twistor program, which hints at a highly category theoretic underpinning. Some preliminary remarks on classical gravity are needed.

3.1 Classical Gravity

Let $I^{-}(p)$ [McC76][Ehr81] be the open set of smooth past-directed timelike curves ending at p in a Lorentzian manifold M. Similarly, define $I^{+}(p)$ for future-directed curves. The causal sets[1] $\{I^{+}(p) \cap I^{-}(q) : p, q \in M\}$ form a basis for a topology on M, the Alexandrov topology.

Lorentzian manifolds equipped with Alexandrov topologies are called *strongly causal*. It is a fact [Ehr81] that two strongly causal metrics g_1 and g_2 on M

[1] these sets are, properly speaking, called *chronological* sets, but we will have no need for a fuller set of definitions

determine the *same* causal structure if and only if the metrics are globally conformal. In other words, the metric of a C^∞ Lorentzian (connected, Hausdorff, paracompact 4-dimensional) manifold M without boundary is determined at a point p, up to a constant, by the light cone in T_pM.

So the homeomorphisms with respect to an Alexandrov topology preserve null geodesics. These homeomorphisms are in fact conformal diffeomorphisms. This recovery of a Lorentzian metric up to a conformal factor from its causal structure is an important motivation behind the causal sets approach to quantum gravity [Lev87][Dow].

In the 1970s a notion of causal boundary was introduced [Pen72] for strongly causal spacetimes, *none* of which are complete in an appropriate sense. Harris [Har00][Sen] has recently considered the categorical structure of topologies on spacetimes with spacelike boundaries.

Lorentzian distance functions differ notably from their Riemannian counterpart. To begin with, it is not necessarily true [Ehr81] that $d(p,q) = 0$ implies that $p = q$. Also, infinite distances $d(p,q) = \infty$ are quite possible. Confining our considerations to non-spacelike separations, the distance is always non-negative. These alterations to the usual axioms for distance functions are, quite remarkably, very like those considered by Lawvere in his classic paper [Law02] relating monoidal categories to generalised metric spaces. By beginning, as we do here, with monoidal structures one is effectively arguing that Lorentzian distance functions are the natural distance functions.

Focusing now on the twistor program of Penrose, we observe that it is the use of sheaf cohomology there that largely motivates our highly abstract investigations later on.

Let x^i for $i = 0, 1, 2, 3$ be coordinates for Minkowski spacetime. Spinor indices [Rin86][Jnr90] are introduced by considering Hermitean matrices

$$\begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix} \equiv x^{AB'} \tag{15}$$

The group $SL(2,\mathbb{C})$ acts on these matrices by conjugation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger \begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{16}$$

where $ad - bc = 1$. This action reduces to the action of the proper, orthochronous Lorentz group on Minkowski space via the two to one cover $SL(2, \mathbb{C}) \to L_+^\uparrow$. Primed and unprimed spinors are interchanged by space or time reflections.

Let e_{AB} and $e_{A'B'}$ be the skew symmetric spinors invariant under $SL(2, \mathbb{C})$. The source free Maxwell equations take the form

$$\nabla^{AA'} \phi_{AB} = 0 \qquad \nabla^{AA'} \widetilde{\phi}_{A'B'} = 0 \tag{17}$$

where the curvature in terms of spinors is

$$F_{AA'BB'} = \phi_{AB} e_{A'B'} + e_{AB} \widetilde{\phi}_{A'B'} \tag{18}$$

More generally, the massless free field equations for particles of helicity $\frac{n}{2}$, in terms of n indices, are

$$\nabla^{AA'} \phi_{AB \cdots L} = 0 \qquad \nabla^{AA'} \widetilde{\phi}_{A'B' \cdots L'} = 0 \tag{19}$$

For example, a massless neutrino is described by a one index ϕ_A. Observe that these equations are actually invariant under the larger conformal group consisting of translations, inversions and dilations.

A compactified version $\mathbb{M}$ of the complexified Minkowski space has the topology of $S^3 \times S^1$. In $\mathbb{CP}^5$ this space is described by the quadric

$$T^2 - V^2 - W^2 - X^2 - Y^2 - Z^2 = 0 \tag{20}$$

associated with $\mathbb{R}^6$ coordinates with metric

$$\mathrm{d}s^2 = \mathrm{d}T^2 + \mathrm{d}V^2 - \mathrm{d}W^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2$$

There is a fourfold cover $SU(2,2)$ of the conformal group which acts on this compactified spacetime. This is the so-called *twistor group*, which acts on $\mathbb{C}^4$ with coordinates given by a spinor pair $(\omega^A, \pi_{A'})$.

The basic twistor correspondence [Rin86] is between points of Minkowski spacetime on the one hand and, on the other, celestial spheres (or rather projective lines) in projective twistor space $\mathbb{CP}^3$. Let $\mathbb{F}_{12}$ denote the flag manifold of pairs (V_1, V_2) for V_i an i-dimensional subspace of the twistor space $\mathbb{C}^4$ and $V_1 \subset V_2$. Then there are projections $\pi_1 : \mathbb{F}_{12} \to \mathbb{CP}^3$ and $\pi_2 : \mathbb{F}_{12} \to \mathbb{M}$.

The coordinates Z of the full twistor space $\mathbb{C}^4$ define null twistors via the condition

$$Z^\alpha \overline{Z}_\alpha = 0$$

and it is these twistors that correspond to spacetime points in the sense that they determine light rays which are incident on the event. By restricting attention to null twistors, there is a map $c : \mathbb{M} \to \mathbb{CP}^3$ taking points to celestial spheres in $\mathbb{CP}^3$. Clearly this is not an ordinary function. In fact, it is our first example of a *span*, being associated to the diagram

$$\begin{array}{ccc} & \mathbb{F}_{12} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{CP}^3 & & \mathbb{M} \end{array} \tag{21}$$

which represents a morphism in a category of spaces.

The twistor correspondence allows one to transform solutions to (19) into Čech cohomology classes. That is, solutions become elements of $H^1(\mathbb{T}^+, \mathcal{S}(-n-2))$ where $\mathbb{T}^+$ satisfies the positive, rather than null, condition and $\mathcal{S}(-n-2)$ is a certain sheaf on $\mathbb{T}^+$.

The natural question is, what happens when we wish to consider massive fields? In the paper [Hur81], Hughston and Hurd combine two solutions to the massless equations for spin $\frac{n}{2}$ particles, thought of as elements of the sheaf cohomology group $H^1(\mathbb{T}^+, \mathcal{S}(-n-2))$. The Klein–Gordon equation solutions for mass m then belong to a second cohomology group $H^2(\mathbb{T}^+ \times \mathbb{T}^+, \mathcal{S}_{m,n}(-\mu - 2, -\eta - 2))$ for $\frac{n}{2} - \frac{1}{2}|\mu - \eta| \in \{0, 1, 2, 3 \cdots\}$. Degree two cohomology arises as a result of the basic Künneth formula [Tu82]

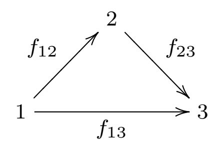
$$H^2(X \times Y) = \bigoplus_{p+q=2} H^p(X) \otimes H^q(Y) \tag{22}$$

relating the cohomology of a product space to the cohomology of the spaces. Naively at least, therefore, a quantisation of this origin of mass involves a non–Abelian sheaf theoretic second cohomology group.

Recall [Tu82] that the first cohomology group is a quotient of cocycles, namely closed forms, by an equivalence relation that says $\omega \sim \sigma$ whenever the forms differ by an exact one. For functions f_{ij} on the intersection of two open sets of the space, the first cocycle condition

$$f_{12} - f_{31} + f_{23} = 0$$

may be thought of as the diagram



Observe that negative numbers in the cocycle condition have been replaced by the orientation of the arrow. Such diagrams make sense in *any* category, as pointed out by Street [Str87], so the coefficients for H^1 may be generalised, in particular to non–Abelian groups.

In other words, categorical cohomology is a powerful framework in which to consider generalisations of the cohomological path integrals which yield exact computations in topological field theory. A motivating example is the rigorous evaluation of Yang-Mills theory in two dimensions [Wit92]. Since categorical dimension will determine the emergent dimension of spacetime, it is expected that cohomology for three and four dimensional categories will play an important role in understanding four dimensional Yang-Mills theory. Some of the special properties of such categories are discussed in chapter 7 and in the appendix. In particular, mass generation is associated to the necessity of breaking the Mac Lane pentagon condition, which is a rule for linear structures such as vector spaces. Thus the forced flatness of space that was recently discovered [Fre] in spin foam quantum gravity is seen as a result of working only with ordinary representation categories. Note also that simple extensions of such spin foam models to braided representation categories do not appear to cure the problem.

Since tetracategorical diagram techniques are as yet entirely non-existent, we begin by investigating amplitudes that may be evaluated using only 1-categorical combinatorics, which will correspond to massless scalar particles.

3.2 MHV Amplitudes

The gluon is a massless boson which, although confined and hence unobservable, will play an important role in high energy experiments at the LHC, and already appears in a new quark-gluon plasma state of matter observed at RHIC.

Incoming and outgoing gluon states are determined by their momenta and their *helicity*, which is basically the handedness of their spin in the direction of motion, labelled plus or minus. In what follows, helicity labels are attached to tree diagrams, and each tree diagram determines an amplitude function of the momenta. A summation over all possible diagrams yields the physical scattering amplitude, just as in ordinary path integral techniques.

A *maximal helicity violating* (MHV) diagram has two negative helicity labels and the remaining labels are positive. Such diagrams have proved useful in analysing amplitudes in perturbative Yang-Mills gauge theory [Wit04a][Wit04b][Svr], where physical Feynman diagrams are constructed using MHV vertices. A propagator internal to a tree diagram, joining two MHV vertices, will have a negative helicity at one end and a positive at the other (see figure 1). MHV evaluations rely on twistor space integrals.

Whereas the gauge theory vertices usually live in Minkowski space, an interaction in twistor space will be localised instead on a celestial sphere $\mathbb{CP}^1$. A momentum p_μ for a massless particle will be expressed as

$$p_{AA'} = \lambda_A \overline{\lambda}_{A'} \tag{23}$$

where the first spinor has positive chirality and the other negative. The spinors contain particle helicity in the sense that a negative helicity fermion wavefunction [Svr] contains a factor λ^A whereas the positive wavefunction contains a $\overline{\lambda}_{A'}$. For the massless particles of spin 1 that we wish to consider the spinor decomposition describes a polarisation, given a helicity ± 1. We therefore consider the gluon data to consist of momenta and helicities. In the next chapter only the signature $++--$ on $\mathbb{C}^4$ will be considered because in that case the spinors are real and hence the categorical dimension is minimised. This signature corresponds to a conformal group of $SO(3,3)$.

Let the two negative helicity indices be X and Y. Then the basic gauge

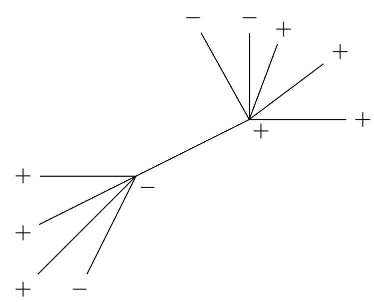


Figure 1: Feynman diagram with MHV vertices

free MHV amplitude for a total of n gluons is defined by [Wit04b][Svr]

$$A_n = g^{n-2} \frac{\langle \lambda_X, \lambda_Y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \tag{24}$$

where $\langle \lambda_1, \lambda_2 \rangle$ is the inner product $e_{AB} \lambda_1^A \lambda_2^B$ and g is the coupling constant.

A standard Fourier transform is used to take this momentum space representation into twistor space. The A_n factor reappears in an expression which localises on a degree 1 curve. For the signature $++--$ this means considering points on a line $\mathbb{RP}^1$ in $\mathbb{RP}^3$.

This result follows from analyticity and so holds for the complex setting of $\mathbb{CP}^1$ in $\mathbb{CP}^3$. It is expected that higher dimensional operads will be useful in computing these complex amplitudes. A general conjecture regarding them [Svr] is that the localisation is on a degree d curve, where at l loops and for q negative helicity gluons

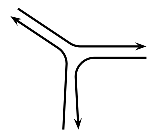
$$d = q - 1 + l$$

Although we only look at the $q = 2$ and $l = 0$ case, it is clear in the operad formalism how to generalise the computation.

3.3 Ribbon Graph M-Theory

For completeness, in this section the main motivation for taking cohomological integrals over moduli spaces is summarised. Such amplitudes may be thought of as background independent path integrals, since the operad interpretation of diagrams lies in an abstract categorical space. One views the cohomological amplitudes as being constrained by the Machian correspondence between the matter and spacetime degrees of freedom, although it is not yet clear how this works in general.

Although this thesis does not deal with dual objects, or categories based on ribbon diagrams, such structures are expected to appear in a fuller treatment of its ideas. A ribbon vertex



is equipped with oriented edges, which define a cyclic ordering on the vertex. Such vertices are used to construct ribbon diagrams in [Pen]. They also appear in 't Hooft's [Sch75] study of large N QCD as three gluon vertices, and in that setting quarks are represented by oriented edges.

The correspondence between Riemann surfaces and ribbon graphs was greatly clarified by Grothendieck, who saw the importance of the Belyi maps. In the next chapter the focus is on simplified gluon amplitudes using twistor techniques for (moduli of) complex surfaces, which are dually related, in Grothendieck's sense, to the ribbon diagrams.

In low dimensions the polytopes of [Batb] describe the n-operad operations. Batanin needed to consider a new compactification of configuration spaces in order to obtain the homotopically correct coherence law table for all n [Dol98]. This means that many of the polytopes for $n \geq 2$ have not yet been applied to the physical question of particle statistics. The original compactification breaks down due to the counterexample of Tamarkin on six points, which is almost certainly of physical interest because it appears in the placement of six gluon points on a $\mathbb{CP}^1$ in twistor space [Wit04a][PFt]. There are three surface moduli that fit into ordinary twistor space $\mathbb{CP}^3$, namely $\mathcal{M}_{0,6}$, $\mathcal{M}_{1,3}$ and $\mathcal{M}_{2,0}$. These moduli are presumably related to the physical massless particle spectrum of gluons, photons and gravitons. Since observables take an entirely new character, there appears to be a clear indication in this theory that extraneous string particles are quite unnecessary.

Observe that the orbifold Euler characteristic [Wal03] of the moduli of the 6-punctured sphere $\mathcal{M}_{0,6}$ equals minus six *which accounts for the three generations of the Standard Model*. This follows from either the usual string theory application of the index theorem or through an interpretation of $-\frac{1}{2}\chi$ as the correct counting of the number of ways to piece together primitive idempotents assigned to punctures on the surface. These idempotents are algebraic representatives [Rio] of special points in projective space which naturally arise in moduli space analysis. Moreover, such an approach was recently used by Brannen [Bra] to both derive the number of generations and to obtain the Koide mass formula for lepton masses. This has since been extended to predictions for the neutrino masses.

In [Pen] Mulase and Penkava give constructive proofs of an isomorphism between the moduli space of a Riemann surface and moduli for ribbon graphs labelled by positive real numbers. These ideas go back to Grothendieck's study of non-singular Riemann surfaces over $\overline{\mathbb{Q}}$. For any such surface Σ there is a Belyi map j from Σ to $\mathbb{CP}^1$. The points $0, 1$ and ∞ are marked on $\mathbb{CP}^1$. Grothendieck considered the inverse of the Belyi map on the interval $[0, 1]$ and showed that the surface Σ may be represented by the resulting ribbons. A triangulation of the surface defines edges and vertices. Thickening the edges to untwisted ribbons gives a basic ribbon graph. The vertices must be of valency at least three.

The category of ribbon graphs is described as follows [Pen]. Let $\mathcal{V}$ be the set of vertices V_i and $\mathcal{E}$ the set of edges of a graph Γ. The ends of each unoriented edge are picked out by the structural incidence map $i : \mathcal{E} \to (\mathcal{V} \times \mathcal{V})/S_2$. The arrows (α, β) in the category are isomorphisms satisfying the commuting diagram

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{i_1} & (\mathcal{V}_1 \times \mathcal{V}_1)/S_2 \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha \times \alpha} \\ \mathcal{E}_2 & \xrightarrow[i_2]{} & (\mathcal{V}_2 \times \mathcal{V}_2)/S_2 \end{array}$$

The full construction requires half edges, just as do Feynman diagrams. A point is simply added to the middle of each edge of Γ to make degree two vertices. At each vertex the set of half edges is then given a cyclic ordering, which describes orientations on ribbon edges. The arrows in a category of such graphs are the arrows just described which preserve the cyclic orderings.

By definition, a *boundary* of a graph Γ is a sequence of directed edges which cycles back on itself. Let b denote the number of boundary components of Γ.

By Euler's relation the genus g of the surface is given by $v - e + b = 2 - 2g$.

The partition function for an interesting matrix model was expressed as an expansion in these ribbon graphs in [Wal03]. The important point is that ribbon graphs are capable of describing the orthogonal, unitary *and* symplectic models. The quaternionic case requires twisted ribbons. Using a few basic graph pieces, all three cases are accounted for by one parameter β which takes on values $1, 2$ and 4 for the real, complex and quaternionic cases respectively. String theoretic T-duality appears naturally in a correspondence between $\beta = 1$ and $\beta = 4$.

Given the natural relation between Jordan algebras [Rio] and projective geometry it is then natural to ask [PFt] whether or not octonionic matrix models can also be described by ribbon graph methods, and whether or not the U-duality of M-theory could be studied in this context. A host of studies on stringy black holes and quantum computation bears testimony to this idea.

The next chapter works towards such an exact gluon phenomenology using $\mathbb{RP}^1 \simeq S^1$ moduli, which are described completely by the Stasheff associahedra 1-operad.

4 Towards Gluon Amplitudes

"And now it goes as it goes and where it ends is Fate."

Aeschylus *Agamemnon*

Around 1960, Regge considered the analytic continuation of scattering amplitudes in terms of angular momenta into the complex plane. Certain pole contributions to the amplitudes, corresponding to quantised momenta, come from bound state particles or resonances. The Regge phenomenology for hadrons, thought of as composites, groups particle poles on linear trajectories which relate mass and spin quantum numbers. In 1968, still before the advent of QCD, Veneziano [Ven68] wrote down a simple expression for a relativistic two particle amplitude in the Regge theory for the strong interactions at high energy. It was the phenomenological success of this Regge theory that led to the development of the original string model for hadron physics.

The interdependence between Regge hadrons, none being truly fundamental particles in this picture, led to a bootstrap hypothesis of self-generation. But since a clear mechanism for the correct classification of particles was lacking, this idea was replaced by the success of the Yang-Mills theory. But meson and baryon labelling comes from the lattice octet and decuplets, fitting onto either a hexagonal or triangular lattice. In the continuum gauge theory context this must be interpreted in terms of the representation theory of $SU(3)$, but having abandoned the continuum we must find an alternative meaning for the Weyl lattices. Fortunately, such lattices arise as binary codes via the inverse image of a code in $\mathbb{F}_2^n$ under the homomorphism from $\mathbb{Z}^n$. This is more of interest in higher dimensions, such as for the $E8$ lattice in $\mathbb{R}^8$, but nonetheless gives computational meaning to hadron classification. In fact, circulant generating matrices for codes are what underlie the mass operators derived in [Bra].

So in modern ribbon graph theory we return to the old Regge ideas, and attempt to derive quantum numbers without the use of representation theory. It is therefore very worthwhile trying to recover the Veneziano amplitudes in this setting.

In this chapter we show that this can be done using operad techniques. For $n \geq 4$, where two particles are regarded as incoming, all n-particle functions are recovered. This suggests that the MHV tree calculus has its origins in ribbon operad physics. In this picture, Veneziano scattering amplitudes are cohomological integrals over moduli spaces for n punctured spheres. For the three punctured Riemann sphere, the moduli space is a point, because up to equivalence there is only one such sphere. In real dimension two, the 3-punctured sphere itself is the moduli for the Veneziano case of four points. By restricting to the real points of this moduli space the simplest integral of this kind is precisely the Veneziano beta function.

In order to see how such integrals are associated to the 1-operad associahedra, we first describe the tiling of the compactified moduli $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ by them [Rea01]. Then expressions for the scattering amplitudes are given in terms of the multiple polylogarithm integrals of [Bro]. A full complex version of this analysis is expected to yield exact n gluon tree amplitudes for QCD, as considered in [Wit04a].

4.1 $\mathbb{RP}^1$ Moduli Spaces

The set of associahedra polytopes, starting with the one point space for the 2-leaved tree, is an operad in the category **Top** of topological spaces. Composition is given by concatenating tree graftings, and this composition is associative since concatenation is. The unit element is the empty space, represented by a single leaf, since grafting a single leaf to another leaf will not increase the number of leaves.

A mapping of this sequence of spaces to another sequence of spaces, which preserves the operad structure, is a morphism of operads. The associahedra appear as cells in the compactified complex moduli $\overline{\mathcal{M}}_{0,n}$, tiling the real points of these spaces. The real sections may be understood independently [Rea01] as moduli of configurations of points on a circle $\mathbb{RP}^1$, an equator of the punctured Riemann sphere. The collection of complex moduli $\overline{\mathcal{M}}_{0,n}$ form an operad, as described below, although there is no natural unit element. This unit problem is of no concern here because the first case of specific physical interest is the four point Veneziano case. There is an alternative extension of this collection of spaces creating a full operad, as shown in [Los04], using two colours for marked points on the spheres. Observe that using two colours for punctures offers a potentially natural means of dealing with helicity labels.

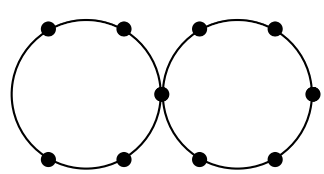
The complex moduli $\overline{\mathcal{M}}_{0,n}$ is the space of isomorphism classes of stable n-punctured genus zero curves. That is, generic elements are Riemann spheres $\mathbb{CP}^1$ and each component will have at least three marked points, in analogy to the trivalency required of ribbon graphs. Stability also means that there may be double points with a neighbourhood isomorphic to the origin of $xy = 0$ in $\mathbb{C}^2$, which are kept separate from the marked punctures. On reducing to the real case, the double points allow unions of circles attached at a common point. In the genus zero case, a curve may be represented by a graph with vertices representing the double points and internal edges the components of the curve, while the external edges are labelled by the punctures.

The operad composition for the moduli operad is thus of the form

$$\overline{\mathcal{M}}_{0,k+1} \times \overline{\mathcal{M}}_{0,n_1+1} \times \cdots \times \overline{\mathcal{M}}_{0,n_k+1} \to \overline{\mathcal{M}}_{0,n}$$

where $n = \sum_i n_i + 1$. Choose one puncture on each curve to be an output, and consider the remainder as inputs. Using the k inputs for $\overline{\mathcal{M}}_{0,k+1}$ as a base, the outputs of the other composition factors are glued to these inputs to create new double points on the resultant genus 0 space. The symmetric group also acts on this operad by permuting factors. This is a *topological operad* since the sequence of moduli lies in the category of topological spaces. Homology, or dually cohomology, on operads gives a new operad of complexes since it is functorial. Higher genus moduli also form (modular) operads.

This gluing of points on complex curves may be mimicked by the $\mathbb{RP}^1$ circles. The uncompactified real moduli space $\overline{\mathcal{M}}_{0,4}(\mathbb{R})$ is defined as the quotient of configurations of points on $\mathbb{RP}^1$ by the action of $PGL_2(\mathbb{R})$. A smooth space is only obtained by adding certain pathological configurations. The Deligne-Mumford compactification for configurations on $\mathbb{RP}^1$ [Rea01] works by adding pieces of moduli that represent the limit of points colliding. These collisions are represented by bubble offshoots of the original $\mathbb{RP}^1$, occurring at a reference point. For example, the diagram



shows five points coming close together in a ten point moduli space. This diagram may alternatively be represented by glued polygons, where the glued edge stands for the point at the intersection of the two bubbles. The pentagon and hexagon on the left of figure 2 represent exactly this bubble diagram.

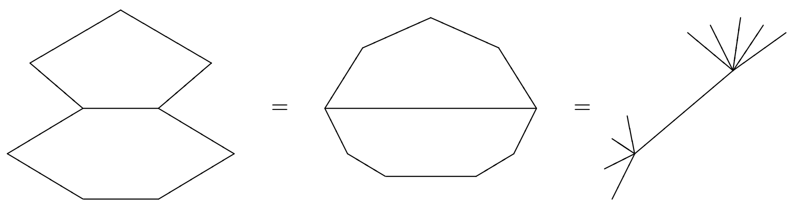


Figure 2: Correspondence of polygons and trees

The tree figure is another alternative, in which the ten edges replace the original points. This brings us closer to the connection between $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ and the Stasheff associahedra that we met in chapter 2. In fact the space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ is tiled (see [Rea01]) by $\frac{1}{2}(n-1)!$ disjoint copies of the associahedron of the same dimension. This shows that the operad of associahedra, as a sequence of topological spaces, lies inside the moduli operad as a suboperad. These polytopes are the key to evaluating the relative cohomology invariants of physical interest [Bro06].

Example 4.1 When $n = 5$ the two dimensional real moduli space is tiled by 12 pentagons. The whole pentagon is represented by the 1-level four leafed tree.

Brown [Bro] considered smooth affine charts on $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ based on the associahedra tilings. Each associahedron cell corresponds to one piece of the chart, so the full moduli spaces may be viewed as a disjoint union of 1-operads. In what follows, it is sufficient to consider only single cells.

For the n-point case, consider an n-gon in the plane with chords (ij). Define coordinates u_{ij} labelled by these chords, given as functions

$$u_{ij} : \mathcal{M}_{0,n} \to \mathbb{CP}^1 \backslash \{0, 1, \infty\}$$

These coordinates are constructed from simplex coordinates as follows. Since $\mathcal{M}_{0,3}$ is a unique point represented by $\mathbb{CP}^1$ with three punctures at $0, 1$ and

∞ the n point case uses only $m = n - 3$ simplex coordinates t_i. By definition [Bro] the u_{ij} are cross ratios

$$u_{ij} \equiv [ii+1 \mid j+1j] = \frac{(z_i - z_{j+1})(z_{i+1} - z_j)}{(z_i - z_j)(z_{i+1} - z_{j+1})} \tag{25}$$

where it is understood that the 3 points z_1, z_2, z_3 are sent to $1, \infty$ and 0 respectively and the remainder are relabeled as t_i. A simplex is defined by a sequence

$$0 < t_1 < \cdots < t_m < 1$$

For the case $n = 5$ there are two coordinates t_1 and t_2 which define a triangular region on the (t_1, t_2) plane. The pentagon tiles of $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ define five chords

$$u_{13} = 1 - t_1 \quad u_{24} = \frac{t_1}{t_2} \quad u_{35} = \frac{t_2 - t_1}{t_2(1 - t_1)} \quad u_{41} = \frac{1 - t_2}{1 - t_1} \quad u_{52} = t_2$$

which defines an embedding in $\mathbb{A}^5$. Let us now look at the first example of a cohomological integral over the pentagon. Let α_{ij} denote concrete hyperplane equations for chords of the form $x_i - x_j$. For integral values α_{ij} the period integral for the five point case is evaluated as

$$I_5 = \int_0^1 \frac{x^{\alpha_{24}}(1-x)^{\alpha_{35}} y^{\alpha_{52}}(1-y)^{\alpha_{41}}}{(1-xy)^{\alpha_{35}+\alpha_{41}-\alpha_{13}}(1-xy)} \mathrm{d}x\mathrm{d}y \tag{26}$$

Observe that such integrals obey a cyclic symmetry due to transposes, which in the five point case take the form

$$T(x,y) = (1 - xy, \frac{1-y}{1-xy})$$

Such symmetries are part of the structure of a cyclic operad, and a known property of the hadron amplitudes under consideration. In general the embedding will be in $\mathbb{A}^{\frac{n(n-3)}{2}}$. Note also that every u_{ij} determines a form $\omega_{ij} = \mathrm{d}\log u_{ij}$ and a formal combination of these is a Knizhnik-Zamolodchikov form [Bro].

Do these integrals form an operad algebra for the moduli operad? In fact, a product for period integrals arises from a Cartesian product decomposition of associahedra. By cutting the planar n-gon into two pieces along a chord (ij), the face of the associahedron defined by the equation $u_{ij} = 0$ may be expressed as the product of two lower dimensional associahedra. For example, cutting a hexagon corresponding to a square face in the three dimensional polytope along its central diagonal divides it into two 4-gons, so this square face of the polytope is given by the product of two 1-dimensional associahedra. Amazingly, this procedure [Bro] induces maps

$$f(n,j,k) : \mathcal{M}_{0,n} \to \mathcal{M}_{0,j} \times \mathcal{M}_{0,k}$$

such that the pullback $f(n,j,k)^*(\omega_j \otimes \omega_k)$ of suitable forms on the lower dimensional spaces gives a product of integrals. The MZV integrals form a rational algebra over a basis set of primitive integrals. Thus an algebra product arises functorially from the decomposition of operad polytope faces.

But is this an algebra in the sense of chapter 2? An operad algebra requires a map into the endomorphism operad of a vector space V. It is not yet clear whether a basis of primitive integrals exists which respects all operad rules, but if one is interested in the product rules, then the tensor product of the integral space with itself gives tensor product of differential forms, which is associated above with the Cartesian product of moduli spaces. The gluing rules for moduli attach punctures to punctures, and a puncture is represented here as an edge of the n-gon. In the example given above, two 3-leaved trees representing squares compose to give a 6-leaved tree representing a hexagon. This indicates that at least one of the MZV product structures arises as an operad algebra.

Note that another motivation for working with this cohomology is that the construction makes sense for curves over any field, but the details require the full machinery of algebraic geometry.

Considering again helicity labels on an MHV diagram, the two negative helicity gluons might be sent to the special points 0 and 1 on the Riemann sphere $\mathbb{CP}^1$. Observe that the MHV tree diagrams are constructed from grafting MHV vertices together, which is a basic operad operation. The grafting process introduces propagators, which correspond to chords on the n-gons. The propagator requirement is simply for the grafted half edges to have negative helicity at one end and positive at the other. The helicity information is not taken into account in what follows, but we note that coloring operads by a small collection of symbols is a standard tool in this kind of combinatorics.

Moreover, since holography suggests replacing the tubular moduli by disc operad diagrams, there is a natural choice of 2-colored operad, which is that used by Jones to describe planar algebras.

4.2 Physical Amplitudes

From the perspective of ribbon graph theory, the real part of an n point function for massless neutral scalar fields should be an integral over a universal cohomology class on the associahedra tiles of the moduli space $\mathcal{M}_{0,n}$. Eventually we would like to consider the full twistor $\mathbb{CP}^3$, which probably entails further categorical levels, namely at least one each for points and lines. In this section we focus on the $\mathbb{RP}^1$ amplitudes, as studied in [Wit04a], since these only require the associahedra integrals outlined above.

In [Bro06] it was shown that such integrals are all given by linear combinations of multiple zeta values [Lew81]. Recall that an MZV of depth d and weight n is a function of the form

$$\zeta(k_1, k_2, \cdots, k_d) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} n_3^{k_3} \cdots n_d^{k_d}} \tag{27}$$

for positive integers $k_1, k_2, \cdots, k_r$ such that $n = \sum k_j$. Such functions regularly occur in QCD amplitudes, along with their further generalisations, the multiple polylogarithms. Higher order polylogarithm functions [Lew81] are defined iteratively via

$$\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(z) \frac{\mathrm{d}z}{z}$$

and in particular $\mathrm{Li}_n(1) = \zeta(n)$. The MZVs form an algebra under two distinct products, but until [Bro06] it was not known how these algebras were related to the associahedra.

In what follows we stick to old-fashioned Regge notation where possible. Thus we let $s_i = (p_1 + \cdots + p_i)^2$ for particle external momenta p_j. As noted above, there will be $n-3$ variables x_j. The running couplings are $\alpha(s_i)$. The period integrals of [Bro] may be reduced to the form

$$B_n = \int_0^1 \prod_{i=1}^{n-3} \mathrm{d}x_i \quad x_i^{-\alpha(s_{i+1})-1} \prod_{1<i<j<n} (1 - x_{i-1} x_i \cdots x_{j-2})^{-p_i p_j} \tag{28}$$

where the $\alpha(s_i)$ and the $p_i p_j$ are integers. The Veneziano type n point functions are precisely of this form [Sch75]. We observe that allowing for ghosts is precisely the same as taking the full form of the Brown integrals. Thus the choice of a ghost cancelling gauge will have a direct interpretation in terms of operad geometry. The four point Veneziano amplitude

$$B_4 = g^2 \frac{\Gamma(-\alpha((p_1+p_2)^2))\Gamma(-\alpha((p_2+p_3)^2))}{\Gamma(-\alpha((p_1+p_2)^2) - \alpha((p_2+p_3)^2))} \tag{29}$$

is given by the beta function precisely because this may be expressed as a period integral in one variable x

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1} \mathrm{d}x \tag{30}$$

An example of the association between MZVs and these elementary period integrals is

$$\int_0^1 \frac{\mathrm{d}x_1 \mathrm{d}x_2}{1 - x_1 x_2} = \zeta(2) \tag{31}$$

which is the dilogarithm evaluated between zero and one. This is an integral on the five point moduli [Bro] and so depends upon two parameters. Similarly for the six point case,

$$\int_0^1 \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{(1 - x_1 x_2 x_3)(1 - x_1 x_2)} = \zeta(3) + \zeta(1,2) \tag{32}$$

Note that $\zeta(3) = \zeta(2,1)$. Observe also that the weight of the MZVs in the integrals above is $n-3$. Brown [Bro] has shown that *any* period integral in the relative cohomology $H^{n-3}(\overline{\mathcal{M}}_{0,n}\backslash A, B\backslash B \cap A)$ for the associahedra suboperad is a rational linear combination of integrals of the type defined above, and given by MZVs of weight at most $n-3$. The integrality of the arguments k_j is being forced by the 1-operad nature of the associahedra. In order to study more general rational arguments for the zeta values one would need to consider the higher categorical operad polytopes.

We would now like to look at the seven point amplitudes. This involves 14 affine coordinates u_{ij} representing the chords of a heptagon. The moduli space for seven points has dimension four and so is tiled by the four dimensional associahedron, whose vertices are labelled by six leaved rooted trees.

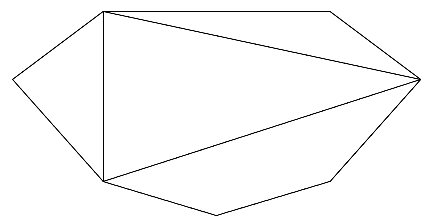


Figure 3: Chorded heptagon for 7 gluons

The example of the figure represents the rooted tree $(2,((3,4),(5,6,7)))$. The associahedron appears as the closure of a region defined by setting $0 < u_{ij} < 1$ [Bro].

The 14 coordinates are

$$u_{13} = 1 - t_1 \quad u_{14} = \frac{1-t_2}{1-t_1} \quad u_{15} = \frac{1-t_3}{1-t_2} \quad u_{16} = \frac{1-t_4}{1-t_3}$$

$$u_{24} = \frac{t_1}{t_2} \quad u_{25} = \frac{t_2}{t_3} \quad u_{26} = \frac{t_3}{t_4} \quad u_{27} = t_4$$

$$u_{35} = \frac{t_3(t_1-t_2)}{t_2(t_1-t_3)} \quad u_{36} = \frac{t_4(t_1-t_3)}{t_3(t_1-t_4)} \quad u_{37} = \frac{(t_1-t_4)}{t_4(t_1-1)}$$

$$u_{46} = \frac{(t_1-t_4)(t_2-t_3)}{(t_1-t_3)(t_2-t_4)} \quad u_{47} = \frac{(t_1-1)(t_2-t_4)}{(t_1-t_4)(t_2-1)}$$

$$u_{57} = \frac{(t_2-1)(t_3-t_4)}{(t_2-t_4)(t_3-1)}$$

Brown introduces cubical coordinates x_i via $t_i = x_i x_{i+1} x_{i+2} \cdots x_{n-3}$. These are the variables appearing in the reduced integrals (28), but we might as well consider the full integral, so

$$B_7 = \int_0^1 \prod_{i=1}^{4} \mathrm{d}x_i \quad x_i^{-\alpha(s_{i+1})-1}(1-x_i)^{\beta_i} \prod_{1<i<j<n} (1 - x_{i-1}x_i \cdots x_{j-2})^{-p_i p_j} \tag{33}$$

where β_i is the usual ghost term, here an integer. Recall that the elimination of ghosts is used in the derivation of the dimension $d = 26$ for bosonic string theory, so these operad techniques offer not just a route to hadron phenomenology, but also the possibility of understanding anomaly cancellation via the principles of the ribbon graph approach. Note that this contact with string theory relies in no way on the standard notions of quantization or observable.

This utility of operad methods in computing physical amplitudes begs the question of a simple categorical statement for the theory. The link between classical logic and twistor gravity also leads one to suspect that a categorical form of gravitational logic is essential. This should reduce to the logic of ordinary quantum physics in the massless regime. The following two chapters are a tentative investigation of braided monoidal categories in this role.

5 Characterisation and Toposes

> *"There is no benefit today in arithmetic in Roman numerals. There is also no benefit today in insisting that the group concept is more fundamental than that of groupoid."*
>
> R. Brown [Bro06]

There are numerous motivations for revisiting, yet again, the foundations of quantum mechanics from a category theoretic point of view. Having looked for similarities between the structure of Minkowski spacetime and properties of massless scalars in QCD, we work now with the assumption that a rigorous framework for QFT may be most succinctly expressed in the operadic language. The explicit correspondence between particle number and spatial dimension is an argument for a new Foch space type of QFT based on monadic annihilation and creation.

The small piece of this picture which is investigated in the following two chapters comes from the simplest imaginable way of extending the elementary axioms for a topos [McL92][Gol84] to a linear realm. The structure is 1-categorical and only quantum mechanical logic is considered. One major motivation for extending classical topos theory is the fact that conjunction (intersection) and disjunction (linearised union) for vector spaces is non-distributive, and can therefore not be accommodated by any classical topos.

If one takes an anti–realist relational view of ordinary quantum mechanics, as is certainly done here, the problem is not so much one of interpretation as a difficulty in finding the correct language with which to describe states. Alternatively, one may choose to abandon the state formalism altogether in favour of an abstract density matrix approach [Bra] in which idempotent operators replace states. This would be more in line with the ribbon graph approach described in chapter 3. However, we stick with the idea that *superposition* should be clearly identifiable in the formalism, but with the hope that the arbitrary normalisation should somehow be removed, and this does appear to be the result.

In this chapter we begin with a discussion of characterisation for categories such as **Vect**.

5.1 Monic Characterisation

The main difficulty in viewing a category such as **Vect** as an elementary topos is that there is no obvious way to introduce the important pullbacks of characterisation. Moreover, since the terminal object is a zero object it appears that any simple attempt to force this axiom on **Vect** will trivialise its topos–like structure, because in a topos, unless trivial, the initial object remains distinct from the terminal.

Every object in **Vect**, with the Axiom of Choice, is isomorphic to an object $F(S)$ for S an object of **Set** since the Axiom of Choice implies that every vector space has a basis. The vector space is therefore isomorphic to the one associated to $F(S)$ where S is the basis set. However, we do not wish to impose the Axiom of Choice on our axioms.

Here we would like to respect the higher categorical nature of monoidal structures, even if this means relinquishing linearity, a property of dubious necessity from a quantum gravitational perspective. The basic structure considered is that of a braided monoidal category. On the other hand, in the domain of ordinary quantum mechanics one would require the recovery of linear structure, so we begin with the observation that the category **Vect** is related to **Set** by an adjunction.

Could this relation be used as a basis for quantum logic? An adjunction is a very fundamental notion in category theory, and **Set** is the model for all classical toposes. Quantum mechanical systems are usually analysed from the perspective of a correspondence principle. Moreover, vector spaces underlie *all* algebras, not only those of proven relevance to physics. Thus we will take as a starting point the axiom that a quantum topos is related to a classical one in this way, leading to the following initial definition.

Definition 5.1 A monoidal category $\Im$ is *characterisable* if there exists an adjunction $(F, G, \eta) : \xi \dashv \Im$ with F a monoidal functor and ξ a topos.

The question still remains as to which class of arrows in $\Im$ should replace the monics in the topos. Since in the case of finite dimensional vector spaces with a basis the pullback property may be preserved for subset monics, it seems natural to choose the least restrictive definition that includes these subspace monics. Moreover, in a 1-dimensional framework the pullback property will still appear ubiquitous in reasonable definitions for the elements of basic quantum logic, so this aspect of characterisation will be maintained.

As a means of weakening the elementary axioms, special classes of monics have been considered in detail by Taylor [Tay] in the context of Stone duality, which will be briefly discussed in chapter 7.

Definition 5.2 A *characterisable monic* in a characterisable monoidal category is a monic $A \rightarrowtail B$ such that there exists a pullback square

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ F(1) & \overset{F(\mathrm{tr})}{\rightarrowtail} & F(\Omega) = \Theta \end{array}$$

for 1 the terminal in the topos ξ and $t = F(\mathrm{tr})$ the fixed image of true.

The arrow

$$F(1) = T \overset{F(\mathrm{tr}) \equiv t}{\rightarrowtail} \Theta = F(\Omega)$$

for $\mathrm{tr} : 1 \rightarrowtail \Omega$ in ξ is monic because tr is a split monic.

Example 5.3 The category **Vect** of vector spaces over a field K with the usual adjunction (defined in chapter 2) from **Set** such that the monoidal functor takes Cartesian product to tensor product. This is a monoidal category with $F(\mathrm{tr}) = K \rightarrowtail K \oplus K$ representing superposition. This is called a *characterising arrow*. $F(\Omega)$ is the *qubit*. The characterisable monics include the ones where selected basis vectors get mapped to basis vectors. $T = K$ is a weak terminal object in **Vect**. The zero object will not appear in the characteristic squares. This example is outlined in more detail at the end of chapter 6.

In a topos with an identity adjunction all monics are characterisable. In a topos T is terminal and there is precisely one characteristic arrow χ for each monic with respect to the characterising arrow tr.

Remark 5.4 Notice that we are potentially restricting our attention to the characterisation of a special class of monics. This idea has been applied directly to a category, without reference to an adjunction [Stra], as a means of weakening the concept of topos. For example, the category **Top** of topological spaces with Cartesian product is not ordinarily characterisable for monics, but is so for the class of closed embeddings, as follows. Let the characteriser $\mathbf{1} \to S$ select the closed point for S the Sierpinski space of one open and one closed point. For a closed embedding $c : X \to Y$, there is a unique arrow to the terminal $\mathbf{1}$ from X. The arrow χ_c must send the image of X to the closed point in S. Given an arrow $Q \to Y$ that satisfies the pullback condition along with the terminal arrow, it must send everything in Q to the image of X. Take $! : Q \to X$ to be the restriction of this arrow to X.

A terminal arrow $! : Q \to 1$ is a natural arrow in a topos because it sits in commutative squares of the form

$$\begin{array}{ccc} Q & \xrightarrow{f} & P \\ {\scriptstyle !_Q}\downarrow & & \downarrow{\scriptstyle !_P} \\ 1 & = & 1 \end{array}$$

and since 1 is terminal the arrow $!_1$ is an identity. Only this condition is dropped on replacing the terminal by the object T in $\Im$.

In the topos **Set** subsets provide monic arrows. The role of monics m in a topos is potentially replaced here by pairs (m, h) of arrows, where h is an arrow into T. It might be natural to think of a *state* as an arrow $m : T \rightarrowtail X$ and a corresponding arrow $h : T \to T$ as a scalar normalisation factor. This idea works only roughly, as we shall see.

5.2 Quantum States

From now on we will be working in $\Im$, always denoting $F(1)$ by T and $F(\Omega)$ by Θ. Let $\mathbf{Sta}(X)$ denote the category whose objects are pairs (m, h), called *states*, where m is a characterisable monic to X and $h \in \Im(sm, T)$ is the unique arrow such that there is an arrow $\chi \in \Im(X, \Theta)$ and the square

$$\begin{array}{ccc} sm & \overset{m}{\rightarrowtail} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \chi} \\ T & \underset{\mathrm{tr}}{\rightarrowtail} & \Theta \end{array} \qquad (34)$$

commutes. This really is just an abstraction of the usual quantum mechanical notion of physical state. For vector spaces over the complex numbers, when $sm = T$ the monic m is a map from $\mathbb{C}$ to the space X, representing a Dirac ket, and the arrow h is a scalar normalisation factor. Since finite vector spaces inherit bases from **Set** via the adjunction, it is unnecessary to introduce independently an inner product structure for state spaces.

An arrow between two states (m, h) and (n, k) is an arrow $\phi : sm \to sn$ such that the diagram

$$\begin{array}{ccc} sm & \overset{m}{\rightarrowtail} & X \\ {\scriptstyle h}\downarrow & \overset{\phi}{\searrow} & \uparrow{\scriptstyle n} \\ T & \underset{k}{\longleftarrow} & sn \end{array} \tag{35}$$

commutes. Observe that ϕ is unique if it exists. We also define a category $\mathbf{Dst}(X)$ of dual (or bra) states over X. This category has as objects pairs (m, k) where $k : T \to sm$ is such that $hk = 1_T$ for an $h \in \Im(sm, T)$ with $(m, h) \in \mathbf{Sta}(X)$. An arrow from (m, k) to (n, l) is an arrow $\phi : sm \to sn$ in $\Im$ such that $m = n\phi$ and $l = \phi k$.

Now assume that $\Im$ has pullbacks. Given an object X let $\mathbf{Sub}(X)$ denote the collection of isomorphism classes of $\mathbf{Sta}(X)$. If (m, h) is a state of X then denote the equivalence class to which it belongs by $[(m, h)]$. Given an arrow $f : X \to Y$ in $\Im$, define a mapping $\mathbf{Sub}(f) : \mathbf{Sub}(Y) \to \mathbf{Sub}(X)$ as follows. The arrow $n \lrcorner f$ in $\Im$ appears in the pullback diagram

$$\begin{array}{ccc} s(n \lrcorner f) & \overset{n \lrcorner f}{\rightarrowtail} & X \\ {\scriptstyle f \llcorner n}\downarrow & & \downarrow{\scriptstyle f} \\ sn & \underset{n}{\rightarrowtail} & Y \\ {\scriptstyle k}\downarrow & & \\ T & & \end{array} \tag{36}$$

for any state (n, k) with $tn = Y$. The monic $n \lrcorner f$ defines a subobject of X where the vertical composition into T gives the l of $[(n \lrcorner f, l)]$.

5.3 Monic Cospan Characterisation

Before fixing upon an initial set of axioms, we consider other alternative characterisations which might be useful generalisations of the topos case. A respect for internalisation invites a barrage of spans and cospans. In categories with pushouts, we would like to characterise with a cospan of the form

$$T \overset{}{\underset{\mathrm{tr}}{\rightarrowtail}} \Theta \overset{}{\underset{\mathrm{fr}}{\leftarrowtail}} F$$

A general monic cospan

$$A \underset{m}{\rightarrowtail} X \underset{n}{\leftarrowtail} B$$

from A to B in $\Im$ will be denoted by $[m, n]$.

In chapter 2 we saw that the cospans of $\Im$ form a bicategory. An arrow from $[m, n]$ to $[p, q]$ is a triplet (f, g, h) of arrows $f : sm \to sp$, $g : tm = tn \to tp = tq$ and $h : sn \to sq$ satisfying the commutative diagram

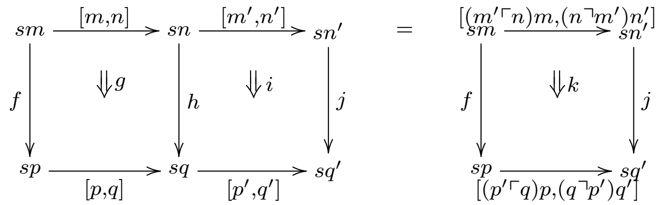
$$\begin{array}{ccccc} sm & \xrightarrow{m} & tm & \xleftarrow{n} & sn \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ sp & \xrightarrow[p]{} & tp & \xleftarrow[q]{} & tq \end{array} \tag{37}$$

Composition is componentwise and the identity arrows are precisely given by all triplets of identities between cospans. Define the support functor **Supp** : **Cospn**($\Im$) $\to \Im$ to be the middle component of an arrow. Explicitly this takes an arrow $(f, g, h) : [m, n] \to [p, q]$ of **Cospn**($\Im$) to the arrow g.

If the category $\Im$ has all binary pushouts then **Cospn**($\Im$) is a double category with vertical arrows given by the arrows of $\Im$, horizontal arrows cospans and objects the objects of $\Im$. A typical square of this double category is given by

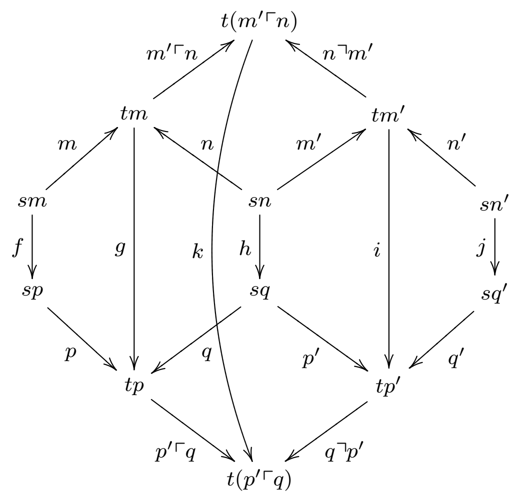
$$\begin{array}{ccc} sm & \xrightarrow{[m,n]} & sn \\ {\scriptstyle f}\downarrow & \Downarrow g & \downarrow{\scriptstyle h} \\ sp & \xrightarrow[{[p,q]}]{} & sq \end{array} \tag{38}$$

and is defined by (37). Vertical composition is as described previously. Horizontal composition of two squares



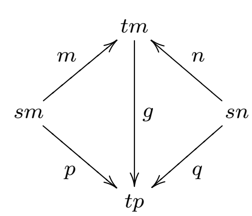
(39)

is given by pushing out cospans as in



(40)

where $k : t(m'^{\ulcorner}n) \to t(p'^{\ulcorner}q)$ is the unique arrow such that $k(m'^{\ulcorner}n) = (p'^{\ulcorner}q)g$ and $k(n^{\urcorner}m') = (q^{\urcorner}p')i$. The bicategory **Cosp**($\Im$) is the double subcategory of **Cospn**($\Im$) restricted to the vertical arrows with identity left and right components. Thus the 2-arrows are commuting diagrams

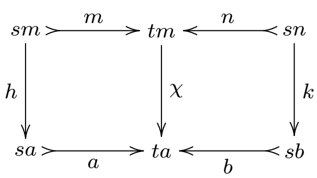


(41)

given by taking $f = 1_{sm}$ and $h = 1_{sn}$ in (37).

So in analogy with the topos definition for single arrow characterisation, for cospans one might consider

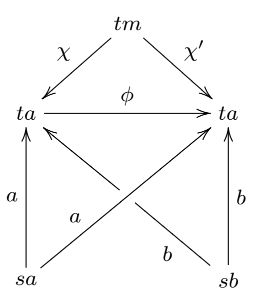
Definition 5.5 A cospan $[m, n]$ in $\Im$ is *characterised* by a cospan $[a, b]$ of monics if for some pair $(h, k) \in \Im(sm, sa) \times \Im(sn, sb)$ there exists a unique arrow χ from tm to ta making the adjacent squares



(42)

a pair of pullbacks.

Definition 5.6 A cospan $[a, b]$ of monics is a *characterising cospan* in $\Im$ if every cospan of monics is characterised by $[a, b]$, and given any cospan of monics $[m, n]$ with h, k and χ as above, and a χ' in place of χ such that the adjacent squares are still pullbacks then there is an arrow $\phi : ta \to ta$ such that all triangles in the diagram

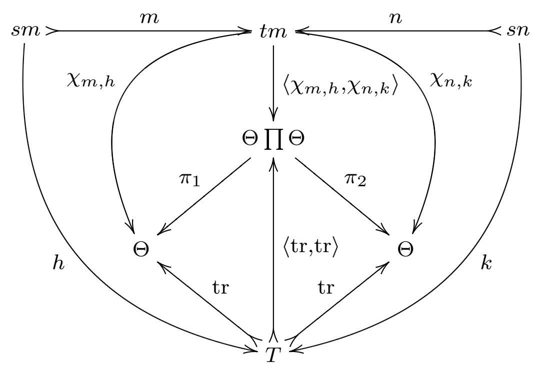


(43)

commute.

Proposition 5.7. *The cospan category of a characterisable category with products and pushouts admits a characterising cospan.*

Proof. Let $\mathrm{tr} : T \rightarrowtail \Theta$ be the characterising arrow in $\Im$. Also, let $\pi_1, \pi_2 : \Theta \prod \Theta \to \Theta$ be the canonical projections of the product. The characterising cospan is then given by $[\langle \mathrm{tr}, \mathrm{tr}\rangle, \langle \mathrm{tr}, \mathrm{tr}\rangle]$ as demonstrated by the following.



(44)

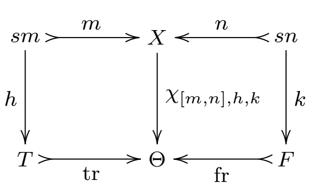
Note that the outside two pieces of the diagram are characteristic squares for m and n. The left half, and respectively the right half, of the diagram thus form a pullback. But this gives a cospan characterisation as required. □

Corollary 5.8. *The cospan category of a topos admits a characterising cospan.*

Now let $\Im$ be a category with characterising cospan

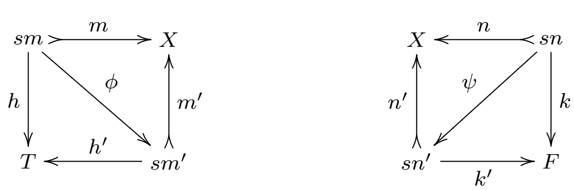
$$T \xrightarrow{\mathrm{tr}} \Theta \xleftarrow{\mathrm{fr}} F$$

Define a cospan subobject functor **Cub** : $\Im^{\mathrm{op}} \to$ **Set** as follows. For X an object of $\Im$, $\mathbf{Cra}(X)$ is defined to be the category of *cospan states* which are triplets $([m, n], h, k)$, where $[m, n]$ is a cospan of monics with $\mathbf{Supp}[m, n] = X$, $h \in \Im(sm, T)$ and $k \in \Im(sn, F)$ along with a characterising arrow $\chi_{[m,n],h,k}$ satisfying the pullback squares



(45)

An arrow from the cospan state $([m, n], h, k)$ to cospan state $([m', n'], h', k')$ is a pair of maps $\phi : sm \to sm'$ and $\psi : sn \to sn'$ such that the diagrams

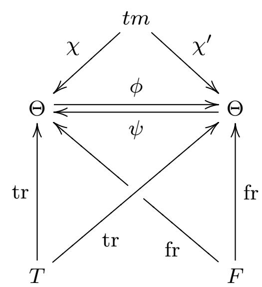


(46)

commute. Define $\mathbf{Cub}(X)$ to be the category of isomorphism classes of $\mathbf{Cra}(X)$. The equivalence class to which a cospan state $([m,n],h,k)$ belongs is denoted $[([m,n],h,k)]$, as before. Given an arrow $f : Y \to X$ we define $\mathbf{Cub}(f) : \mathbf{Cub}(X) \to \mathbf{Cub}(Y)$ by

$$\mathbf{Cub}(f)[([m,n],h,k)] = [([m\lrcorner f, n\llcorner f], h(f\llcorner m), k(f\lrcorner n))] \tag{47}$$

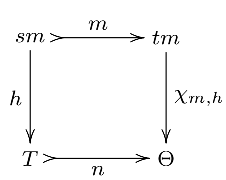
Now define the cospan characteristic functor $\mathbf{Car} : \Im^{\mathrm{op}} \to \mathbf{Set}$ by $\mathbf{Car}(X) = \Im(X,\Theta)\backslash \sim_X$ where this time $\chi \sim_X \chi'$ if and only if there are arrows $\phi, \psi \in \Im(\Theta,\Theta)$ such that all six triangles in the diagram



(48)

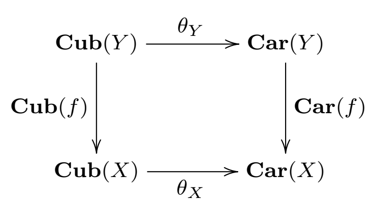
commute. Given an arrow $\chi \in \Im(X,\Theta)$ the equivalence class to which it belongs is denoted by $\langle\chi\rangle$. Given an arrow $f : Y \to X$ we define $\mathbf{Car}(f)\langle\chi\rangle \equiv \langle\chi f\rangle$.

There is a natural isomorphism $\theta : \mathbf{Cub} \Rightarrow \mathbf{Car}$ as follows. It is sufficient to concentrate on one half of the cospan diagrams. Given a state (m,h) there exists $h \in \Im(sm,T)$ and $\chi_{m,h} \in \Im(tm,\Theta)$ such that



(49)

is a pullback square. We define $\theta_X[(m,h)] = \langle\chi_{m,h}\rangle$. It must be shown that the natural square



(50)

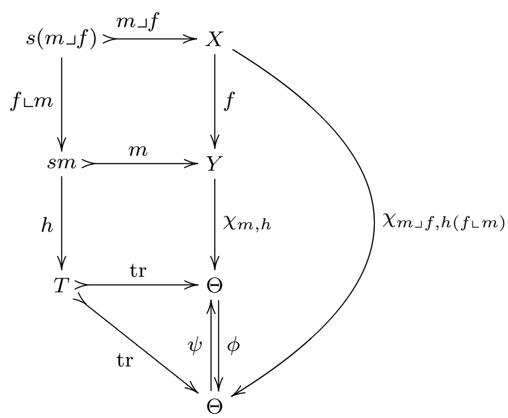
holds for all arrows $f : X \to Y$. Chasing the subobject corresponding to a state (m, h) with $tm = 1_Y$ around the square,

$$\mathbf{Car}(f)\theta_Y[(m,h)] = \mathbf{Car}(f)\langle\chi_{m,h}\rangle = \langle\chi_{m,h}f\rangle \tag{51}$$

Alternatively, we have

$$\theta_X\mathbf{Cub}(f)[(m,h)] = \theta_X[(m\lrcorner f, h(f\llcorner m))] = \langle\chi_{m\lrcorner f,h(f\llcorner m)}\rangle \tag{52}$$

where $\chi_{m\lrcorner f,h(f\llcorner m)} : X \to \Theta$ are arrows such that the outside of the following diagram is a pullback.



(53)

There exist arrows $\phi, \psi \in \Im(\Theta, \Theta)$ such that all triangles and squares in the above diagram commute. Hence $\langle\chi_{m\lrcorner f,h(f\llcorner m)}\rangle = \langle\chi_{m,h}f\rangle$ and it follows that θ is natural.

The definition 5.3 of cospan characterisation is extendable to any number of characterising arrows tr_i in an obvious way. Cospans would be replaced by larger families. In particular, a topos may be characterised by n copies of the monic arrow $\langle \mathrm{tr}, \cdots, \mathrm{tr}\rangle$ for the n-fold product, since in the diagram 44 we saw how two characteristic squares in a topos could be turned into a cospan characterisation, and with all finite products the arrow $\langle \mathrm{tr}, \mathrm{tr}\rangle$ into $\Theta \prod \Theta$ may be replaced by the n-fold version.

The example 2.3 suggests the intriguing idea that one might want to consider logics with multiple basic truth types. In topos theory there is only one type of truth, whether the category is Boolean, two–valued or otherwise. That is, characterisation need only be described with respect to truth. This fact goes hand in hand with the ubiquity of posets, which are simply categories enriched in the category **2** of two objects and one non–identity arrow.

Lawvere's [Law02] natural description of *distances* as truth types points to a possible need for a lot more than one or two! Alternatively, we could *weaken* the notion of truth. As it happens, this alternative appears to be the more natural one.

5.4 Definition of a Quantum Topos

In utilising the notion of weak characterisation via adjunction to define a quantum topos we have in mind *both* the model of the category of vector spaces and a category of categories **Cat**, to be used in analogy with the role of **Set** as a model for toposes.

Direct sum of vector spaces is a categorical product, but the product structure is not closed. Therefore, the first alteration to the topos axioms is to replace the categorical product with a monoidal one. Bear in mind the example of Cartesian product in **Set** being replaced by ordinary tensor product of vector spaces. Also note that although the examples considered have terminal objects, they do not play the same important role as in a topos, and are absent from the following axioms. The necessity of the braiding will become clear when we investigate some of the consequences of these axioms.

One aim here is to avoid a forcing of the symmetry axiom, as truly braided examples are of some interest. Thus there must not be too many restrictions on the functors F and G comprising the adjunction. For example, assuming that G is monadic would place us in the situation of the algebraic theories of Lawvere [Law69], since there exists an adjunction between a topos ξ and the monoids in ξ precisely when ξ has a natural numbers object, such as does **Set**. Then ε might equalise the symmetry as in the diagram

$$FGA \otimes FGB \xrightarrow{\varepsilon_{A\otimes B}} A\otimes B \underset{1_{A\otimes B}}{\overset{\gamma_{BA}\gamma_{AB}}{\rightrightarrows}} A\otimes B$$

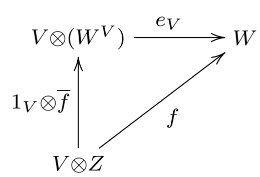
which is the case if F preserves a certain kind of equaliser.

Definition 5.9 A *quantum topos* is a characterisable braided monoidal category $\Im$ which has

1. a monoidal product $\otimes : \Im \times \Im \to \Im$ with unit T, a left (right) unit constraint λ_V (ρ_V) and braiding arrows γ_{UV}

2. the characterising adjunction $F \dashv G$ with data $(F, G, \eta, \varepsilon)$, with F monoidal, for $G : \Im \to \xi$ with ξ a topos, and the monic image of truth in ξ, $\mathrm{tr} : T \to \Theta$

3. (*right closure*) exponential objects (W^V) defined by the commuting transpose diagram, for any arrow $f : V \otimes Z \to W$,



4. for any object V, the arrow $\gamma_{\Theta^V V} \cdot \gamma_{V \Theta^V}$ is a right unit for e_V

Monoidal closure follows from the right closure axiom in the presence of a braiding. Observe that there exists at least one arrow from any object into T. That is, T is weakly terminal. If $\otimes$ is the categorical product with unit T then T is a terminal object and the axioms are clearly satisfied by a topos for symmetric braiding.

Example 5.10 The category **Vect** of vector spaces over a field K with tensor product (and direct sum). The truth arrow maps the field K to the qubit $K \oplus K$.

Observe that since the corresponding category of Hilbert spaces and bounded linear maps is not closed it will not satisfy the axioms. Until a higher categorical framework is developed, closure appears essential to the fundamental theorem, discussed below. It would be interesting to develop Hilbert space structures more abstractly using inner products derived from the bra and ket structures considered here. That is, it appears to be more natural to derive a category of Hilbert space objects from the underlying quantum topos axioms, rather than trying to force the structure of the usual category on the chosen axioms.

Example 5.11 The category $\mathbf{Rep}_G$ of all linear representations of a group $\mathbf{G}$ with tensor product and equivariant linear maps. It is symmetric monoidal closed. There are a number of ways to define this category. Firstly, it is the category of vector space objects in the category **GSet** of functors from $\mathbf{G}$, viewed as a one object category, into **Set**. **GSet** is a Boolean topos, which may be used in an adjunction with $\mathbf{Rep}_G$. Any functor X in **GSet** may be composed with the adjunction functor F from **Set** into **Vect** to obtain a linear representation of $\mathbf{G}$. Conversely, any group representation V, which is a functor from $\mathbf{G}$ into **Vect**, defines an object of **GSet** by composition with the forgetful functor of the adjunction between **Vect** and **Set**. There is a Yoneda embedding of $\mathbf{G}$ into **GSet** which is full and faithful, and describes representations by composition with F.

Example 5.12 The Cartesian product $\Im_1 \times \Im_2$ of two quantum toposes in a category of categories. The characterising arrow is the arrow $(\mathrm{tr}_1, \mathrm{tr}_2)$ since monics and pullbacks are preserved under Cartesian product. The functors $\otimes_1 \times 1_{\Im_2 \times \Im 2}$ and $1_{\Im_1 \times \Im 1} \times \otimes_2$ in conjunction with the universal property of Cartesian product define a functor $\otimes$ into $\Im_1 \times \Im_2$ as required. Finally, there is an adjunction between this product category and $\xi_1 \times \xi_2$, the product of the toposes in the axioms, since Cartesian product takes pairs of arrows (F_1, F_2) and (G_1, G_2) for the adjunction maps and the required natural transformations exist since $1_{\Im_1 \times \Im_2} \simeq 1_{\Im_1} \times 1_{\Im_2}$ by the assumed functorality of Cartesian product. In a more thorough bicategorical setup, this example could be weakened.

5.5 Monoidal Structure

A monoidal structure with unit e provides an action of the commutative monoid $\Im(e, e)$ on each $\Im(X, Y)$. A *scalar* is by definition an arrow $h \in \Im(T, T)$ that belongs to some $(m, h) \in \mathbf{Sta}(tm)$ for a monic m of source T. Denote the action $\hat{\_}$ of the monoid $\Im(T, T)$ on $\Im$. That is, for an arrow $f : A \to B$ the action of $h : T \to T$ on f is given by the composite

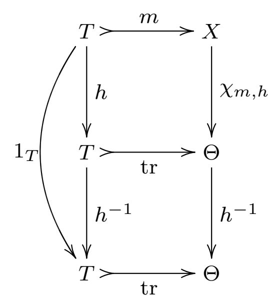
$$A \xrightarrow{\rho_A^{-1}} A \otimes T \xrightarrow{f \otimes h} B \otimes T \xrightarrow{\rho_B} B$$

The monoid is commutative in the sense that $hk = kh$ for scalars h and k, since $\rho_T = \lambda_T$. Let $\Im^*(T, T)$ be the invertible elements of $\Im(T, T)$.

For any characterisable monic m of source T, whenever h is invertible for a state (m, h) one has that $(m, 1_T)$ is a state. Moreover, considering the action of $\Im^*(T, T)$ on characteristic arrows, one has

$$\chi_{m,h}\hat{k} = \chi_{m,kh}$$
$$\chi_{mk,h} = \chi_{m,h}k^{-1}$$

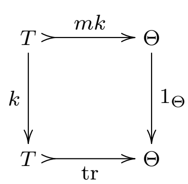
The first statement follows simply from the composition of the two squares



and we observe that

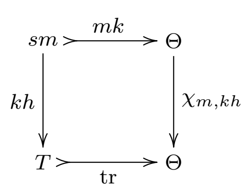
$$\chi_{m,T} = h^{-1} \cdot \chi_{m,h}$$

In other words, for such states the non-zero scalars $h \in \Im^*(T, T)$ act simply as normalising values. The square



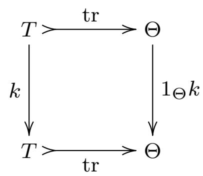
(54)

is characterising if k is invertible. For observables m of source distinct from T, the diagram



(55)

follows from the composition of the characteristic square for (m, h) with the pullback

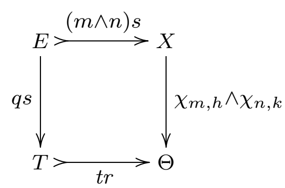


Thus there is a well defined action of $\Im^*(T,T)$ on generalised states. The existence of scaling is another demonstration that the normalisation factors may be ignored in a topos like setting for ordinary quantum logic.

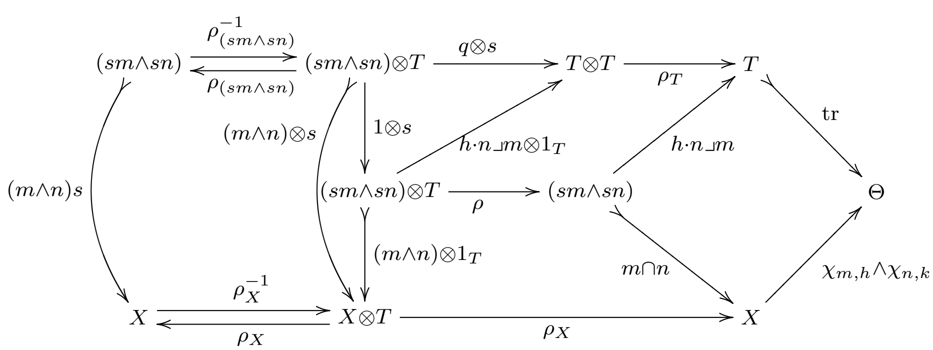
Take subobjects $[(m,h)], [(n,k)] \in \mathbf{Sub}(X)$.

Definition 5.13 The *meet* of $[(m,h)]$ and $[(n,k)]$ is defined by a characterising square for the pullback $m \wedge n$ of m and n.

Lemma 5.14. *The action of $s \in \Im^*(T,T)$ on the meet state $[(m \wedge n, q)]$ of $[(m,h)]$ and $[(n,k)]$ results in a characterising square*



Proof. The outside of the diagram



gives the commutativity of the square as required. □

The arrow $\chi_{m,h} \wedge \chi_{n,k}$ will be discussed properly in the next chapter. Consider the case when the states have pullback characterising squares. Then take two arrows from Q into X and T on the right hand side of the diagram such that when tensored with 1_T we may apply the pullback $(sm \cap sn) \otimes T$ to obtain a unique arrow from Q. Hence there is an arrow $Q \to (sm \wedge sn)$. Finally, since the left hand square is a pullback, this arrow $Q \to (sm \wedge sn)$ is unique.

It follows that

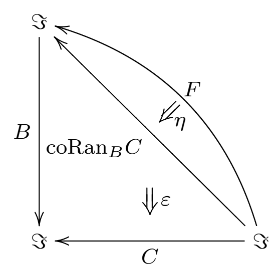
$$(m \wedge n)s = ms \wedge ns \tag{56}$$

and since the action of $\Im^*(T, T)$ preserves equivalence classes of states, it forms an action on subobjects.

Now we consider how monoidal exponentials can be expressed in terms of Kan extensions.

Definition 5.15 A Kan coextension $\mathrm{coRan}_B C$ of a functor $C : \Im \to \Im$ along a functor $B : \Im \to \Im$

(57)



is a functor $\mathrm{coRan}_B C$ as shown along with a natural transformation ε such that given any $F : \Im \to \Im$ and $\overline{\eta} : FB \Rightarrow C$ there is a unique η satisfying

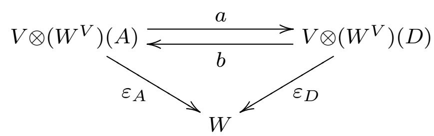
$$\overline{\eta} = \varepsilon \cdot B\eta$$

that is, ε has the universal property.

For axiom 4 the functor B becomes $V \otimes \_$ and the functor C is the constant functor sending all arrows to the identity on W. The natural transformation ε gives the evaluation arrows for the exponentials and the functor $\_ \otimes (W^V) = \mathrm{coRan}_{V \otimes \_} W$. Thus the diagram of axiom 4 becomes a coextension upon categorification, as for a monoidal category as a one object bicategory. The four objects become functors, the three arrows become natural transformations.

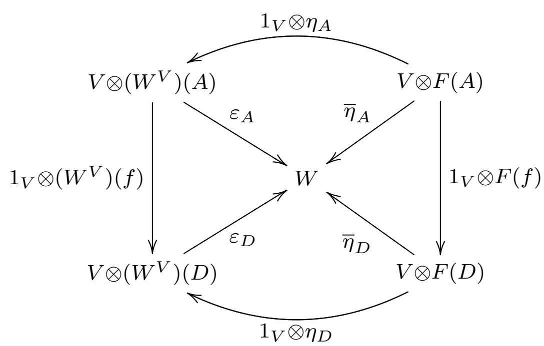
Proposition 5.16. *A category $\Im$ with bifunctor $\otimes$ has coextensions $coRan_{V \otimes \_} W$ for all objects V and W if and only if it is (right) closed with respect to $\otimes$.*

Proof. Given coextensions, closure follows from the correspondence just described. Now let us assume right closure. Consider F to be the constant functor sending everything to Z then, given arrows $f : A \to D$ and $\overline{\eta} : V \otimes Z \to W$ in $\Im$, the vertical arrow of the transpose diagram exists as the one component of the natural transformation η given an exponential W^V indexed by A or D. However, it is always true that $(W^V)(A) \simeq (W^V)(D)$ because in the diagram



(58)

one may take $a = 1_V \otimes \overline{\varepsilon_A}$ and $b = 1_V \otimes \overline{\varepsilon_D}$. Since the respective triangles commute, these arrows provide the required isomorphism. It remains to consider alternative choices for $(F, \overline{\eta})$, as in the diagram



(59)

□

6 The Logic of a Linear Quantum Topos

"If one believes that the comprehension scheme is a basic ingredient in mathematical thought, then the entire theory presented here is already rigidly determined."

J. W. Gray [Gra69]

In this chapter logical operations are derived from the stated axioms for a category such as **Vect**, which has products. The main difference with ordinary topos logic is the weakening of the notion of truth, which is motivated by the example of **Vect**. All concepts introduced reduce to the topos theoretic ones under the relevant restrictions. This is a requirement of this approach. The logic of **Vect** should also reflect the physical meaning of bra and ket states in ordinary quantum mechanics.

Because an initial object plays no important role, the notion of *falseness* must be re–evaluated. In fact, it is only possible to define falsity at all in a quantum topos with more than one monic into T. Fortunately, this is true for **Vect** as it has a zero object.

Rules of inference [Sco86] in categorical logic are steps or trees of proofs (arrows) built from trivalent vertices. These are written as in the example

$$\frac{L \vdash \phi \qquad\qquad L \vdash \psi}{L \vdash \phi \wedge \psi}$$

where L denotes a collection of hypotheses that entail the expression on the right of the $\vdash$ symbol. In a category, the components of the tree are arrows and axioms that relate arrows are expressed as equivalence relations between logical proofs. Propositions appearing in such trees are taken to have truth values, that is, they are arrows into the object Θ in the quantum topos. For example, the existence of exponentials with respect to Θ may be expressed by the rule

$$\frac{V \otimes W \to \Theta}{W \to \Theta^V}$$

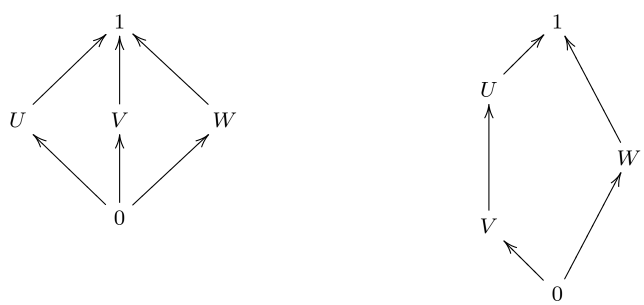
Rules for internal logic will be addressed in the final section of the chapter.

6.1 Lattices

Lattice theory is central to the propositional calculus of quantum mechanics [Pir76]. A lattice is a poset with 0 and 1 and binary operations $\wedge$ and $\vee$. Traditionally, quantum logic [Kal83] is characterised by lattices equipped with an orthocomplement, the rules for which force Boolean logic in a topos.

In ordinary quantum mechanics one is interested in the lattice of subspaces of a Hilbert space. Inclusion of subspaces acts as an order relation $\leq$ which in a categorical context is thought of as a directed arrow. An orthogonal subspace $\neg U$ of a subspace U of a Hilbert space $\mathcal{H}$ satisfies the complement rules. Intersection represents $\wedge$, and $U \vee W$ is the smallest subspace of $\mathcal{H}$ containing the union of U and W. These operations are associative and commutative. The distinction between $\vee$ and set theoretic union breaks distributivity. This shows that quantum logic has no chance of being described by topos theoretic lattices, which are always distributive.

A lattice is distributive [Kal83] if and only if it does not contain closed sublattices



(60)

For example, for subsets $A \subset B$ with C disjoint from B and A, the union $B \cup C$ is outside the lattice. On the other hand, given a two dimensional subspace U, in a three dimensional vector space, containing a line V, along with a two dimensional subspace W not containing V, one obtains a poset like the pentagon (see figure) which is not distributive.

An orthomodular lattice is characterised [Kal83] by the rule

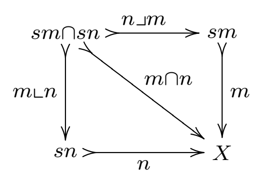
$$\text{if } U \leq V \text{ and } V \wedge \neg U = 0 \text{ then } U = V \tag{61}$$

If $U \leq V$ means that $U \wedge \neg V = 0$ as it does for sets, this condition says that $U \wedge \neg V = V \wedge \neg U = 0$, or rather that $U \leq V$ as well as $V \leq U$, so orthomodularity automatically holds in this case.

Rather than start off by worrying about the lattice structures of quantum mechanics [Coe02] we take the view that these should be derived from a more foundational axiomatics, in analogy with the elementary characterisation of toposes.

6.2 Subobjects and Meets

In a topos, subobjects of X are just monics m into X and it is clear that the equivalence classes of such monics form a meet semilattice with top element 1_X and with the meet $[m] \wedge [n]$ defined by the intersection pullback of m along n, denoted $sm \cap sn$, as in



Here, meets $[(m,h)] \wedge [(n,k)]$ must determine another class in $\mathbf{Sub}(X)$. Fortunately, $\Im$ has all binary products and pullbacks, so it has all equalisers.

Recall that the pullback of m along n is denoted $n \lrcorner m$ and similarly for $m \llcorner n$.

Lemma 6.1. *Let $e : E_{m,n} \rightarrowtail sm \cap sn$ be the equaliser of $h \cdot n \lrcorner m$ and $k \cdot m \llcorner n$. The meet of $[(m,h)]$ and $[(n,k)]$, defined by*

$$[(m,h)] \wedge [(n,k)] \equiv [((m \cap n)e, (h \cdot n \lrcorner m)e)]$$

is associative.

Proof. Let $P_{(mn)q}$ be the pullback of $E_{m,n}$ and sq. Similarly, define $P_{m(nq)}$. By the universal property of the equaliser $E_{m,n}$ there is an arrow $P_{(mn)q} \rightarrow E_{m,n}$ such that the appropriate diagram commutes. Thus we may apply the pullback property of $P_{m(nq)}$ to obtain an arrow $P_{(mn)q} \rightarrow P_{m(nq)}$. An identical argument works in reverse. This sets up an isomorphism $P_{(mn)q} \simeq P_{m(nq)}$. Note that $e = 1_{sm \cap sn}$ when T is terminal, so this definition reduces to the usual one when $\Im$ is a topos. We must verify that the meet is well-defined on equivalence classes. Consider another element (m', h') of $[(m,h)]$. Then there is an arrow

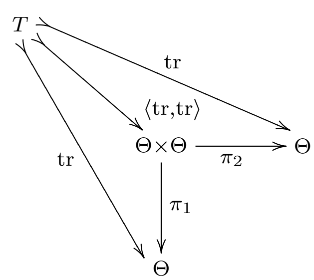
$\phi : sm \to sm'$ such that $m' \cdot \phi \cdot n{\lrcorner}m = n \cdot m{\llcorner}n$. Using the pullback property of $m' \cap n$ there is an arrow $m \cap n \to m' \cap n$. This arrow satisfies the equaliser condition for e' defined with respect to m' and n. Thus there is an arrow $E \to E'$ as required. □

We observed in the last chapter that the monic 1_X is an upper element for the category $\mathbf{Sub}(X)$. The subobject $[(m, k)]$ is always contained in $[(1_X, h)]$. The rule $[(m, h)] \wedge [(m, h)] = [(m, h)]$ is retained.

6.3 Conjunction and Monoidal Conjunction

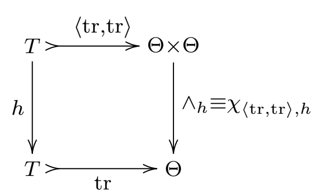
Given both a monoidal structure and pullbacks there are two possible definitions of conjunction, based on $\Theta \otimes \Theta$ and $\Theta \times \Theta$ respectively. Although these objects may be isomorphic, the definitions still differ.

Define $\langle \mathrm{tr}, \mathrm{tr} \rangle$ as in the diagram



(62)

Then ordinary conjunction is the characteriser in

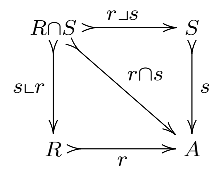


(63)

where we let $\wedge \equiv \wedge_h$.

Focusing momentarily on conjunction in relation to the object $\Theta \times \Theta$, consider the following. Fix two monics $r : R \rightarrowtail A$ and $s : S \rightarrowtail A$. Let $h_r \in \mathbf{Sta}(r)$ and $h_s \in \mathbf{Sta}(s)$ be characterised by χ_{r,h_r} and χ_{s,h_s} respectively.

The intersection $R \cap S$ is defined by the pullback

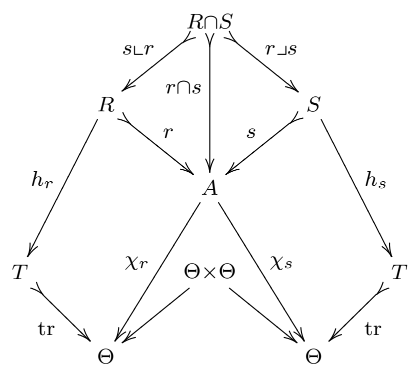


(64)

The condition

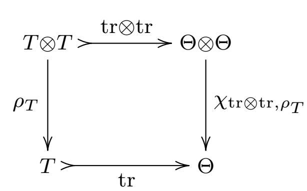
$$h_s \cdot r_{\lrcorner}s = h_r \cdot s_{\llcorner}r \tag{65}$$

is now required to use the pullback property of $\Theta \times \Theta$ in the diagram



to obtain an arrow $\langle \chi_r, \chi_s \rangle : A \to \Theta \times \Theta$. This condition is equivalent to $\chi_s(r \cap s) = \chi_r(r \cap s)$ by monicity of tr.

For the monoidal case, consider the characterising arrow of the monic tr⊗tr, which is monic by axiom 2, given by the pullback



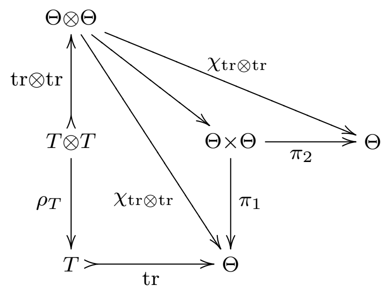
(66)

This will be used to construct a monoidal conjunction.

Lemma 6.2. *There exists an arrow* $!(h) : \Theta \otimes \Theta \to \Theta \times \Theta$ *such that*

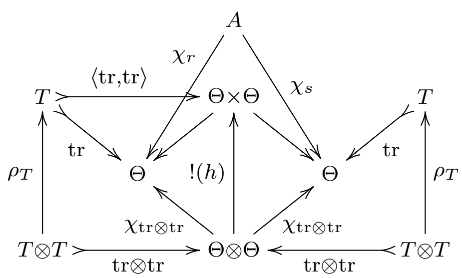
$$!(h) \cdot (tr \otimes tr) = \langle tr, tr \rangle \cdot \rho_T$$

Proof. Follows from the diagram

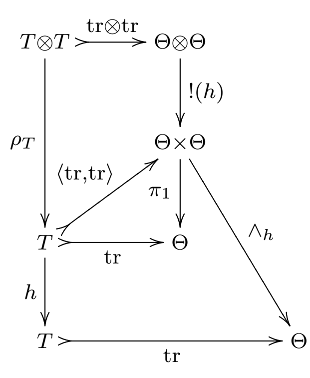


□

Observe that it does *not* follow from the diagram



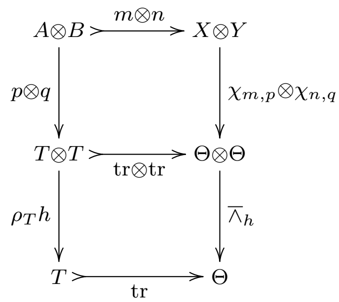
that there is an arrow $A \to \Theta \otimes \Theta$. However, the natural choice for monoidal conjunction is to define $x \overline{\wedge}_h y$ as the composition of $\overline{\wedge}_h \equiv \wedge_h \cdot !(\rho_T)$ with the arrow $x \otimes y : X \otimes Y \to \Theta \otimes \Theta$, allowing for the sources to differ. Observe that $\overline{\wedge}_h$ characterises $\mathrm{tr} \otimes \mathrm{tr}$ using $h\rho_T : T \otimes T \to T$, as shown by the diagram



(67)

in which the upper trapezium is a pullback since the square is, recalling that $\chi_{\mathrm{tr}\otimes\mathrm{tr}} = \pi_1 \cdot !(h)$.

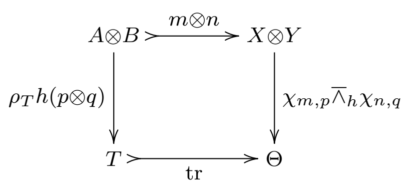
Thus the composition of two pullbacks



gives a notion of

$$\chi_{m,p}\overline{\wedge}_h\chi_{n,q} \equiv (\chi_{m,p} \otimes \chi_{n,q})\overline{\wedge}_h \tag{68}$$

in a characteristic pullback



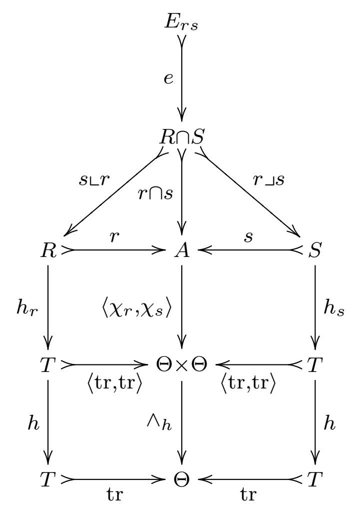
(69)

Here we have used the fact that the top square is a pullback by decomposition of each side into the components, giving two possible unique arrows into $A \otimes B$ which together restrict to one.

Observe that the tensor product of states in $\mathbf{Sta}(X)$ preserves subobject classes under the choice $\phi \otimes \phi'$ of 1-arrow between classes.

Proposition 6.3. $\chi_{r,h_r} \wedge_h \chi_{s,h_s}$ *characterises* $(r, h_r) \wedge (s, h_s)$.

Proof. In the diagram, the arrow e is the equaliser of the meet. One can take the choice $hh_r \cdot s \llcorner r \cdot e \in \mathbf{Sta}(r \wedge s)$ for the left hand arrow of a characterising pullback square, as follows. Note that $\chi_r \wedge_h \chi_s$ is the composition of the bottom two central vertical arrows. Let Q be any object of $\Im$ such that there are two arrows $Q \to T$ and $Q \to A$ making the outer diagram commute, where the first arrow has the lower left T as target. By composition with $\langle \chi_r, \chi_s \rangle$ one can apply either the lower left or lower right pullback to obtain a unique arrow $Q \to T$ into the upper left T. Now use this arrow to apply either of the left or right hand pullbacks to obtain unique arrows $Q \to R$ and $Q \to S$.



Finally, use the pullback property of the intersection to show that there exists a unique arrow $Q \to R \cap S$. □

Now let $\mathbf{Sub}(\Im)$ denote the category whose objects are the thin categories $\mathbf{Sub}(X)$ labelled by objects of $\Im$. The arrows are the $\mathbf{Sub}(f)$ as defined above. Also, let $\mathbf{Char}(\Im)$ be the category of comma objects $(\Im(X, \Theta), \Theta)$ quotiented by the given equivalence relation.

Observe that on $\mathbf{Sub}(\Im)$ there is a monoidal structure

$$\mathbf{Sub}(X) \otimes \mathbf{Sub}(Y) = \mathbf{Sub}(X \otimes Y)$$

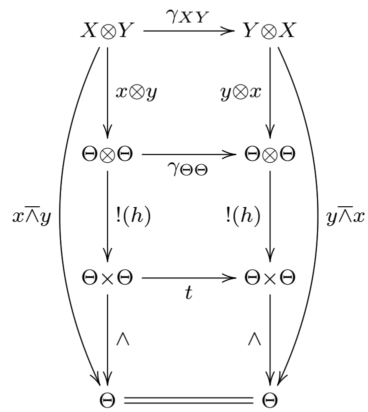
and on $\mathbf{Char}(\Im)$ a monoidal structure

$$\mathbf{Char}(X) \overline{\wedge} \mathbf{Char}(Y) = \mathbf{Char}(X \otimes Y)$$

In fact, this establishes the following result.

Proposition 6.4. *The natural isomorphism $\theta : \mathbf{Sub} \Rightarrow \mathbf{Char}$ is a monoidal natural transformation between monoidal functors.*

These definitions raise the question of the role played by the braidings $\gamma_{XY} : X \otimes Y \to Y \otimes X$ in $\Im$. Quantales [Ros03] involve noncommutative conjunctions, such as one finds here in diagrams such as

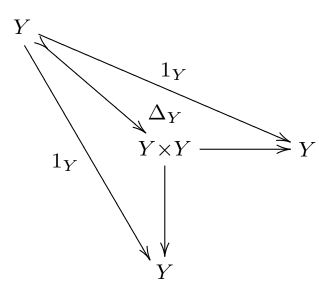


which shows that $(x \overline{\wedge} y) = (y \overline{\wedge} x)\gamma_{XY}$.

Example 6.5 In **Vect** over $\mathbb{C}$, for scalars $x, y : \mathbb{C} \to \mathbb{C}$ one has relations $(x \overline{\wedge} y) = q(y \overline{\wedge} x)$ for q the scalar γ_{TT}. This is usually known as the *quantum plane*.

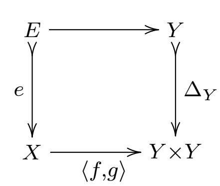
6.4 Implication

The definition of implication in a topos uses the equaliser of conjunction and the projection $\pi_1 : \Theta \times \Theta \to \Theta$. This construction may be followed here. The diagonal map is defined via the pullback diagram



(70)

Observe that Δ_Y is monic. As in a topos [Moe92], an equaliser of two arrows $f, g : X \to Y$ is the monic arrow e in the pullback



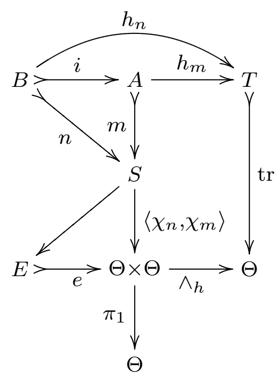
(71)

where $\langle f, g \rangle$ is the unique arrow of the pullback $Y \times Y$ for the arrows f and g. This definition of equaliser is applied to the arrows π_1 and $\wedge_h$.

For two monics $m : A \to S$ and $n : B \to S$ we would like to consider the relation between the condition

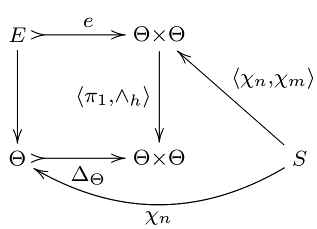
$$\chi_m(h_m) = \chi_m(h_m) \wedge_h \chi_n(h_n)$$

and the equaliser e, although technically e is no longer a *relation* in the right sense. The condition appears in the diagram



(72)

for which the pullback property for A gives an inclusion monic i representing $n \subseteq m$. Alternatively, consider the arrow $S \to E$ arising from the pullback property for E as in the diagram



(73)

where the condition

$$\langle \pi_1, \wedge_h \rangle \cdot \langle \chi_n, \chi_m \rangle = \Delta_\Theta \cdot \chi_n$$

is as given. This establishes

Proposition 6.6. *$\chi_m(h_m) = \chi_m(h_m) \wedge_h \chi_n(h_n)$ implies that $n \subseteq m$ and $\langle \chi_n, \chi_m \rangle \subseteq E$ for any choices of $h_m \in \mathbf{Sta}(m)$ and $h_n \in \mathbf{Sta}(n)$.*

The *material conditional* is then a characteristic arrow for the monic e as in the square

$$\begin{array}{ccc} E & \overset{e}{\rightarrowtail} & \Theta\times\Theta \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \rightarrow_u} \\ T & \underset{\mathrm{tr}}{\rightarrowtail} & \Theta \end{array} \tag{74}$$

For subobjects (r, h_r) and (s, h_s) of an object A any arrow classified by the arrow $\chi_r \rightarrow_u \chi_s$, such as that generated by the pullback with tr, is known as a *material implicate*.

6.5 Universal Quantification and Bra States

In analogy with [McL92], but for monoidal product, for an object X of $\Im$ we would like to define an arrow $\overline{\mathrm{tr}_l}$ to be the transpose of

$$X\otimes T \xrightarrow{\rho_X} X \xrightarrow{l} T \overset{\mathrm{tr}}{\rightarrowtail} \Theta \tag{75}$$

Definition 6.7 An object X is *universally quantifiable* if there exists an arrow $l : X \to T$ such that the transpose $\overline{\mathrm{tr}_l}$ of the arrow (75) is monic. Such an l is called a *bra state*.

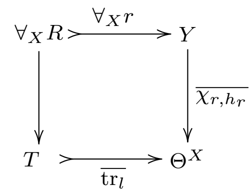
In general, bra states for a monic m of source X define a subcategory $\mathbf{Bra}(tm)$ of $\mathbf{Sta}(tm)$, where the latter is a *thin category*, meaning it has at most one arrow between non–isomorphic objects. The introduction of bra and ket states emphasises the importance of the physical Dirac notation.

The existence of scale factors allows us to restrict our attention to the scalar 1_T. The universal quantifier for X arises as the characteriser of the pullback

$$\begin{array}{ccc} T & \overset{\overline{\mathrm{tr}_l}}{\rightarrowtail} & \Theta^X \\ {\scriptstyle 1_T}\downarrow & & \downarrow{\scriptstyle \forall_{X,l}} \\ T & \underset{\mathrm{tr}}{\rightarrowtail} & \Theta \end{array} \tag{76}$$

Definition 6.8 A *relation* is a monic arrow $r : R \rightarrowtail X \otimes Y$.

Given a relation $r : R \rightarrowtail X \otimes Y$ with characteristic χ_{r,h_r} for $h_r \in \mathbf{Sta}(r)$ the object $\forall_X R$ is the pullback

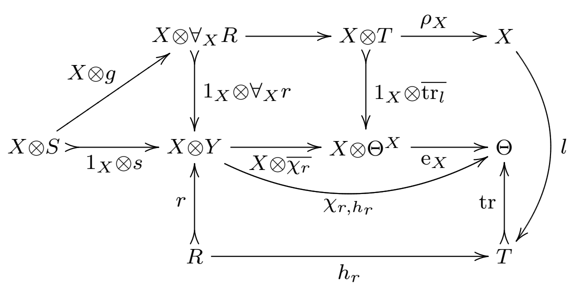


(77)

The following applies to quantum toposes such as **Vect**, where T is not terminal.

Proposition 6.9. *For a monic $s : S \rightarrowtail Y$, $S \subseteq \forall_X R$ implies that $X \otimes S \subseteq R$. If l is right invertible, then $X \otimes S \subseteq R$ implies $S \subseteq \forall_X R$.*

Proof. Recall that monics are preserved by tensor product. Tensoring the pullback above with X gives the diagram



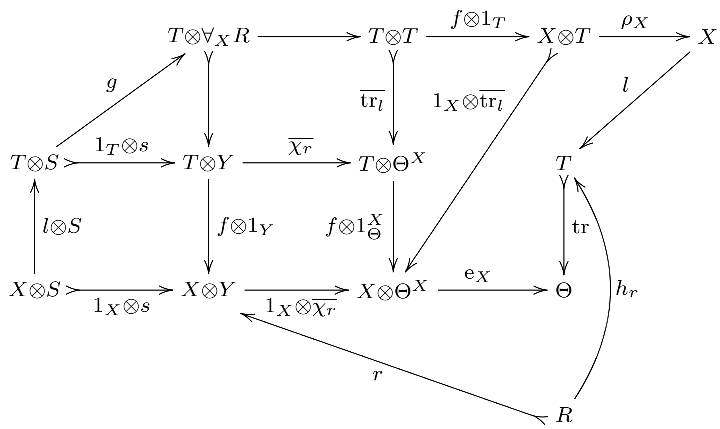
for a given monic g representing containment in $\forall_X R$. By the pullback property of R there is a unique arrow $X \otimes S \rightarrowtail R$, clearly monic as required. Conversely, if l was right invertible, then there would be an arrow $T \to X \otimes T$ on the right–hand side of the diagram. Coupled with an arrow $X \otimes S \rightarrowtail R$ representing the inclusion, this pair satisfies the pullback condition for $X \otimes \forall_X R$. □

Note that in the case where $X = T$ we have $\Theta^T \simeq \Theta$ and $e_T = \rho_{\Theta^T}$, and so for $f : T \otimes T \to \Theta$ the arrow $\overline{f}$ provides the inverse necessary to make the proposition work both ways.

In a topos there are *no* conditions on the proposition and the converse also holds [McL92] because a terminal T easily allows arrows into $T \otimes T$ in the diagram. For a topos [Moe92] one considers quantifiers in terms of adjoints to the functor that pulls back elements of $\mathbf{Sub}(Y)$ along the projection $X \times Y \to$

Y. This projection arrow becomes an arrow $z : X \otimes Y \to Y$ playing no clear analogous role.

Alternatively, consider the diagram



in which it is assumed that there exists an arrow $f : T \to X$ such that

$$X \otimes s = (f \otimes Y)(T \otimes s)(l \otimes S) \tag{78}$$

and where the interchange law for arrows in a monoidal category is used. This 2-dimensional piece of the structure of a bicategory will appear in chapter 6. Given the monic g, one may apply the pullback R to the arrows from the object $X \otimes S$ to obtain an arrow as required. This arrow is monic by composition. Conversely, given an arrow $i : X \otimes S \to R$ there is an arrow $\mathrm{tr} \cdot h_r \cdot i$ into Θ. This enables us to use the pullback property of $\forall_X R$ if there exists an arrow $w : T \to T \otimes T$ such that $w = (l\rho_X(f \otimes T))^{-1}$. In categories where l is invertible, and ρ is the identity, simply take $f = l^{-1}$. In this case, the condition (78), albeit a strong one, may replace that of the above proposition and then the implication works both ways, as it does in a topos.

6.6 Internal Semantics

In a linear space quantum topos such as **Vect**, one views an element $\psi \in V$ as a physical entity. Sentences in the language are therefore physical questions about combinations of states. Internalisation allows us to think about this language in much the same terms as the syntax [McL92].

Variables are associated to their *type*, an object of $\Im$. Words are formed from variables and arrows in $\Im$. In particular, given a word a of type A and a variable b of type B there is always an associated word of type A^B, which is called a *bound* word.

Consistency in logic is about the truth of interpreted statements. In a topos [McL92] one considers arrows $A_1 \times \cdots \times A_n \to \Theta$ to be *interpretations* of a truth valued statement ϕ with respect to variables $\{x_1, \cdots, x_n\}$ belonging to $A_1 \times \cdots \times A_n$. The pullback

$$\begin{array}{ccc} \Phi(\underline{x}) & \longrightarrow & T \\ \downarrow & & \downarrow \mathrm{tr} \\ A_1 \times \cdots \times A_n & \xrightarrow[\phi(\underline{x})]{} & \Theta \end{array} \tag{79}$$

defines the *extension* $\Phi(\underline{x})$ of the statement ϕ. This seemingly esoteric language allows us to give meaning to statements such as $X = Y$. In this case, the extension $[X = Y](\underline{x})$ appears in the equaliser

$$[X{=}Y](\underline{x}) \rightarrowtail A_1 \times \cdots \times A_n \underset{Y(\underline{x})}{\overset{X(\underline{x})}{\rightrightarrows}} \Theta \tag{80}$$

which forces $X(\underline{x}) = Y(\underline{x})$ as arrows. This shows, in particular, that $X = X$. The important set theoretic axiom, in local form, that follows from this is the comprehension axiom, which in a topos takes the form: for any statement ϕ, $\vdash \underline{x} \in \Phi(\underline{x}) = \phi$. This is more familiarly phrased as: *given a property ϕ there exists a set of elements for which ϕ holds.*

Now we wish to discuss concrete states of a *quantum* system in a framework in which they at least exist mathematically. But in the monoidal case the above notion of semantics must be adjusted. As a first step, one might simply replace the product by $A_1 \otimes \cdots \otimes A_n$, for we do indeed think of variables as elements of such combinations of state spaces. This is the operational approach that shall be used in the next section.

However, in addressing the comprehension axiom itself, sets need to be replaced by categories. It was Gray [Gra69] who first considered the categorical comprehension scheme. In showing that it was possible to formulate it, he was led to a careful study of the 2-categorical nature of **Cat**, thereby heralding the development of higher dimensional algebra. In the future, a deeper analysis of quantum toposes should develop internal semantics that respect the full comprehension scheme.

6.7 The Fundamental Theorem

The fundamental theorem of topos theory shows that a comma category (ξ, X) of a topos is also a topos. Moreover, the functor $f^* : (\xi, Y) \to (\xi, X)$ preserves exponentials and the object Ω. Is there an analogue of this theorem for quantum toposes? The adjunction between $\Im$ and a topos ξ certainly yields an adjunction between comma categories $(\Im, X)$ and (ξ, GX). However, we would like to consider the details of the classic topos theorem in order to understand a higher dimensional analogue, if it exists.

Here we will simply explore what the basic quantum axioms tell us, which is that there is indeed an analogue to the fundamental theorem for certain universally quantifiable objects. Our approach is probably closest to that of McLarty [McL92]. All the axioms are needed in the analogue to the fundamental theorem, which is seen as an interesting weakening of the core lemma of the topos case. There is also an additional property required, characterising certain objects in the category.

Definition 6.10 Objects A and B of $\Im$ are *entropic* if there exist monic arrows $j_{AB} : A \times B \rightarrowtail A \otimes B$ and similarly j_{BA} such that the following diagram commutes.

$$\begin{array}{ccc} A \times B & \overset{j_{AB}}{\rightarrowtail} & A \otimes B \\ {\scriptstyle t_{AB}} \downarrow & & \downarrow {\scriptstyle \gamma_{AB}} \\ B \times A & \underset{j_{BA}}{\rightarrowtail} & B \otimes A \end{array} \tag{81}$$

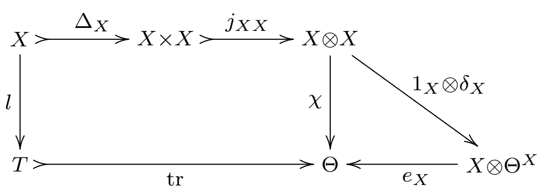
We consider entropic objects of the linear quantum topos $\Im$. Observe that due to the existence of a diagonal $X \rightarrowtail X \times X$ this amounts to a consideration of *classical objects* in the sense that there is a way for the object to faithfully copy itself. A similar idea was studied recently in [Pavb].

Any object in an ordinary topos is classical and in quantum mechanics the observer interacts via collapse onto a classical state. For quantum gravity observables it may be necessary to relax the assumption that measurement happens through collapse, so we would like a clearer understanding of the process. It will turn out that for linear quantum toposes, Θ is a classical object. Thus ordinary quantum mechanics, expressed in terms of categorical formulas, may be thought of as a collapse of possible statements into the truth values of the classical object.

This concept of collapse has the advantage of being local in spacetime, since it occurs in some sense at a single event, represented by the entire topos. Recall that the basic twistor correspondence replaces a spacetime point with a celestial sphere, or projectivised $\mathbb{C}^2$. This $\mathbb{C}^2$ is exactly the object Θ in the quantum topos **Vect**. So classical objects for measurement provide a direct link between quantum mechanics and causality for massless systems.

In this section it is shown that the object Θ is always a natural classical object in the sense that it retains important properties of objects in a classical topos.

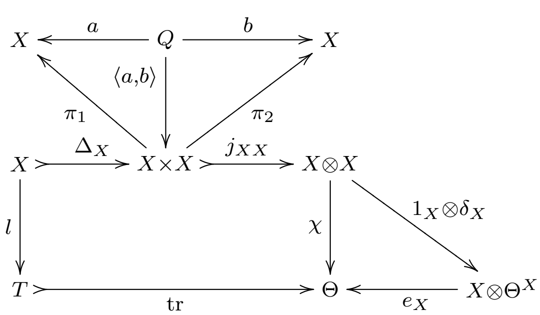
Let $\delta_X : X \to \Theta^X$ be the transpose of the characteriser χ_{Δ_X} appearing in



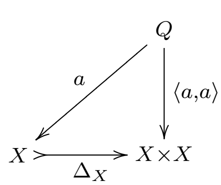
(82)

Lemma 6.11. *For arrows $a, b : Q \to X$, $a = b$ if and only if $tr \cdot la = \chi \cdot j_{XX}\langle a, b\rangle$.*

Proof. Consider the diagram



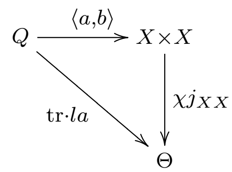
for any arrows a and b. If $a = b$ then one may add the triangle



to the diagram, showing that

$$\mathrm{tr} \cdot la = \chi \cdot j_{XX} \langle a, b \rangle$$

Conversely, if this expression holds then $\pi_2 \Delta_X a = b$, showing that $a = b$. In other words, the triangle



commutes if and only if $a = b$. □

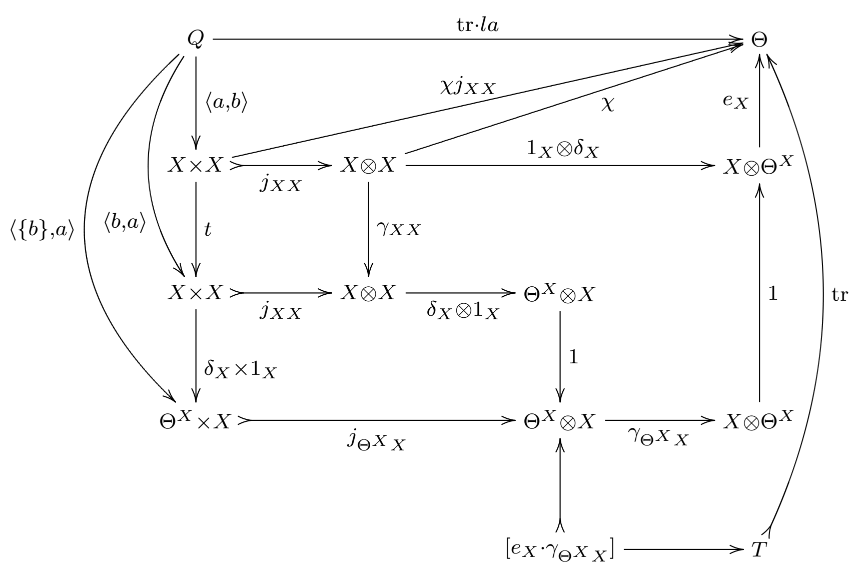
The fundamental theorem requires the construction of a characteriser based on any object X of $\Im$, which uses the following concept [McL92].

Definition 6.12 Given any arrow $b : Q \to X$, the *singleton* arrow $\{b\}$ is given by $\{b\} = \delta_X \cdot b$. The statement $a \in \{b\}$ is the arrow $e_X \gamma_{\Theta^X X} \cdot j_{\Theta^X X} \langle \{b\}, a \rangle$. We say that $a \in \{b\}$ is *true* if it factors through the extension $[e_X \cdot \gamma_{\Theta^X X}]$.

This choice of nomenclature is clarified by the following lemma.

Lemma 6.13. *$a \in \{b\}$ is true if and only if $a = b$.*

Proof. The extension pullback appears in the lower right of the diagram



Looking at the outside of the diagram, the pullback condition is met if and only if $a = b$, as required. □

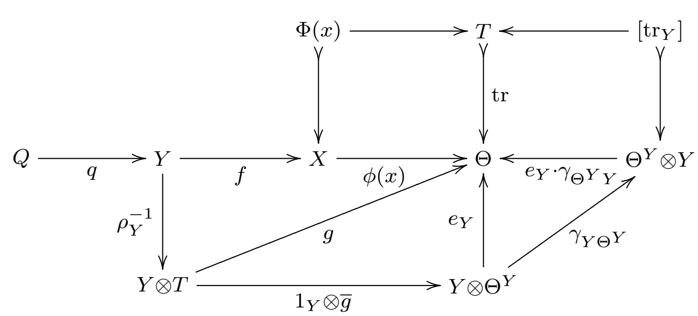
The appearance of singletons is related to the interpretation of characterisation as a type of partial function property [Gol84] in the sense of recursive function theory. Replacing T by an object X, one would like to extend the so called domain of definition of $h : sm \to X$ to tm. This is done by replacing all $x \in X$ by the singleton $\{x\} \in \Theta^X$. For **Set** it is sufficient to define an object $[\Phi_X]$ as the set of all $\{x\} \in \Theta^X$ together with a marker object $\{0\}$, where 0 is the initial object of **Set** [Gol84]. We will need to consider a $[\Phi_X]$ for any quantum topos. This should be thought of as the extension representing the statement, for variables ϕ of type Θ^X, *for any* $x \in X$, $x \in \phi$ *implies that* $\phi = \{x\}$.

Definition 6.14 For an extension $\Phi(x)$ over an object X and a particular arrow $q : Q \to X$ we say that $\Phi(q)$ *is true* if there exists an arrow $Q \to T$ so that the pullback property holds and thus q factors through $\Phi(x)$.

Having replaced Cartesian product by monoidal structure in the definition of boundedness for variables, one would like to consider the following.

Definition 6.15 [McL92] Given arrows $f : Y \to X$ and $q : Q \to Y$, $\Phi(fq)$ is true if and only if q factors through $[\mathrm{tr}_Y] \equiv [e_Y \gamma_{\Theta^Y Y}]$.

The pullback conditions for the two extensions are linked via the diagram



in which axiom 5 appears at the lower right, and g by definition is the composition shown.

Observe how it really is possible to replace the topos theoretic twist maps by braiding arrows in the quantum case. These ideas reduce to the usual ones when monoidal structure is just Cartesian product. The following argument illustrates how for the linear quantum topos the full fundamental theorem only works when the monoidal structure is precisely the Cartesian product, as in a topos. It demonstrates that the comma category only behaves nicely when $X = \Theta$, unless the monoidal structure is precisely the Cartesian product. However, this does not rule out weaker results of interest that may follow from the axioms. The added conditions include the following definition, on which a few remarks will shortly be made.

Definition 6.16 An object X of $\Im$ is *doubly universally quantifiable* if

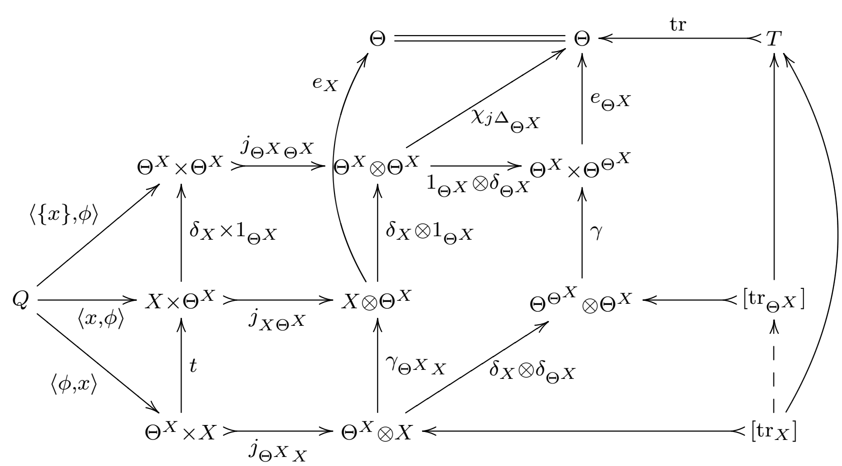
1. X is universally quantifiable with respect to the bra state $l : X \to T$
2. there exists a bra state $k : \Theta^X \to T$ such that $l = k\delta_X$

In a topos this is true for all objects X since the arrows concerned are simply terminal arrows. The statement of interest here is then the following. Given a doubly universally quantifiable object X in a linear quantum topos $\Im$, such that $X \times X \simeq X \otimes X$, there exists an object $[\Phi_X]$ in $\Im$ and an arrow $s_X : X \rightarrowtail [\Phi_X]$ which is characterising for monics in $(\Im, X)$. Moreover, the pullback of an arrow $x : Q \to X$ with this arrow is an exponential Θ^x in the comma category.

What follows is more by way of a clarification than a rigorous proof of what must be a rather difficult theorem. The question is, what special object $[\Phi_X]$ to choose so that we may apply the previous lemma. One choice for a topos [McL92] is the extension for the formula, for ϕ of type Θ^X,

$$\forall_X((x \in \phi) \to (\phi = \{x\}))$$

Consider the lemma 6.13 on ϕ and $\{x\}$ as variables of type Θ^X, which depends on the arrow $\langle\{x\}, \phi\rangle$ as in the diagram

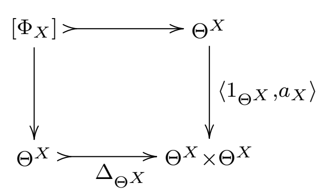


where the top path represents the statement $\{x\} \in \{\phi\}$. The mysterious triangle involving the evaluation e_X depends on the condition of the theorem, as shown below.

If $\{x\} \in \{\phi\}$ is true there is an arrow $q : Q \to T$ such that the pullback condition on $[\mathrm{tr}_{\Theta^X}]$ holds. But then the pullback condition on $[\mathrm{tr}_X]$ holds, with respect to the statement that $\phi = \{x\}$ using the predicate of equality.

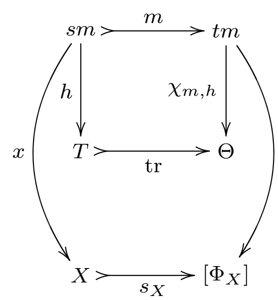
It turns out that the monicity of the arrow δ_X is a difficult question. Here we simply observe that since G preserves monics, the arrow $\chi_{G\Delta}$ will be monic in the topos ξ. Recall that this arrow appears in the characteriser by application of FGF, but this is only monic if F has a left adjoint. Moreover, composition with $\omega_{X \times X}$ and $\varepsilon_{F(\Theta)}$ does not necessarily preserve monicity. In other words, a better generalisation of the topos δ_X would require a different set of axioms, perhaps relying on a triple adjunction string. Here we simply assume that these conditions are met, as they are for finite dimensional vector spaces in **Vect**.

The special object $[\Phi_X]$ appears, by definition, in the equaliser pullback



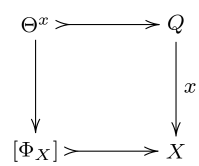
where a_X is defined as follows. Since δ_X is monic, the arrow $\langle 1_X, \delta_X \rangle$ into $X \times \Theta^X$ is also monic. a_X is the transpose of a characteriser χ for the composition of this arrow with $j_{X\Theta^X}$. Now there is an arrow $s_X : X \rightarrowtail [\Phi_X]$ by application of the equaliser property, and s_X is monic as the first factor in a composition defining the monic δ_X.

By the factoring lemma, given an arrow $x : Q \to X$ there is an arrow $Q \to [\Phi_X]$. Thus s_X is characterising in $(\Im, X)$, since for any (m, h) with arrows $x : sm \to X$ and $y : tm \to X$ there is a factor arrow into $[\Phi_X]$ from tm.

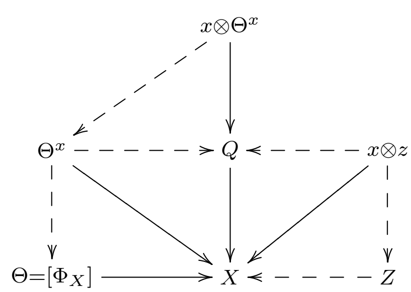


To use the pullback property in $\Im$, one needs the arrow $l : X \to T$. Choose $h = lx$.

For any object $x : Q \to X$ of $(\Im, X)$, define the object Θ^x via the pullback



Similarly, define an object $x \otimes z$ for any object $z : Z \to X$. These comma category objects appear as solid arrows in the diagram

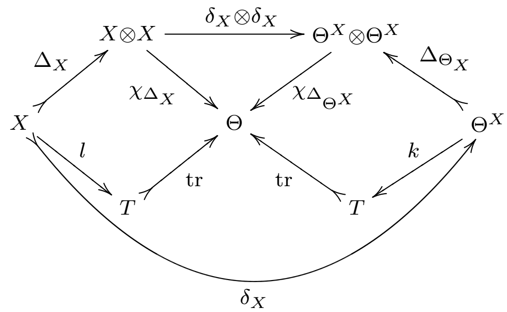


the left hand side of which is a new evaluation.

The crucial dependence of the above argument on the condition that j_{XX} be an isomorphism appears in the triangle containing e_X. This triangle requires, assuming that j_{XX} is an isomorphism, that

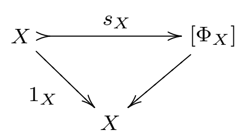
$$\chi_{\Delta_X} = \chi_{\Delta_{\Theta X}}(\delta_X \otimes \delta_X)$$

using the transpose diagram for δ_X. This condition follows from



so long as $l = k\delta_X$, as is assumed.

Finally, let $\Im$ be a quantum topos with pushouts and coproducts. In this case, for an object X satisfying the conditions considered above the comma category $(\Im, X)$ is a kind of quantum topos in the following sense. The identity arrow 1_X is terminal in $(\Im, X)$, so as a quantum topos it has a terminal object, which is useful in yielding, for instance, factorisation. Products $f \times g$ are defined by pullback. One might choose the arrow $\pi_2 : \Theta \times X \to X$ as a subobject classifier in $(\Im, X)$ [Moe92]. However, the arrow



of the theorem allows us to set $T = 1_X$. In other words, $(\Im, X)$ behaves like a topos. Define the arrows j_{AB} to be identities between the two equivalent monoidal structures. $(\Im, X)$ is right monoidal closed, since given an exponential with respect to Θ all others exist via the usual topos theoretic proof using the existence of a terminal object, pullbacks and pushouts [Moe92].

Remark 6.17 The linking of the arrows k and l may be further studied through a consideration of higher adjoints in the sense of pseudomonads [Str04]. That is, given a natural transformation arrow $\varepsilon : \Theta^{\Theta^X} \to X$ the composition

$$g \equiv (1_{\Theta^X} \otimes \overline{\mathrm{tr}_k})(1_{\Theta^X} \otimes l)(\delta_X \otimes \varepsilon)$$

naturally appears in the square

$$\begin{array}{ccc} X\otimes\Theta^{\Theta^X} & \xrightarrow{\;g\;} & \Theta^X\otimes\Theta^{\Theta^X} \\ {\scriptstyle 1_X\otimes\Theta^g}\downarrow & & \downarrow{\scriptstyle e_{\Theta^X}} \\ X\otimes\Theta^X & \xrightarrow[e_X]{} & \Theta \end{array}$$

of internalising composition. This operates at the level of elementary axioms for a category of categories.

In the case of **Vect**, the only object to satisfy the conditions of the theorem is the object of truth values Θ. So only the comma category $(\Im, \Theta)$ results in a linear topos in the obvious way. Moreover, since T is terminal the monoidal structure is like Cartesian product. In other words, the comma category $(\mathbf{Vect}, \Im)$ looks a lot like the topos **Set**. It would be especially interesting to pursue the question of the analogue of Stone duality for sets [Moe92]. In **Vect** over $\mathbb{C}$ the double power set for Θ becomes the collection of maps $f : \mathbb{C}^2 \to M(2, \mathbb{C})$ from the qubit to the two–by–two matrices over $\mathbb{C}$, or $\mathbb{C}^4$. On projectivisation, these become maps $\mathbb{CP}^1 \to \mathbb{CP}^3$, which as we have seen are of interest in ordinary twistor theory as representations of classical spacetime events and, moreover, appear in ribbon graph M-theory with further structure.

The canonical reduction of classical objects in **Vect** to a Boolean topos is therefore interpreted as saying that the language of quantum truth–valued propositions will be collapsed in a measurement situation.

6.8 Truth and Falsity for Vector Spaces

In a quantum topos initial and terminal objects do not necessarily play an essential role. Consequently, the NOT operation $\neg$ should be interpreted slightly differently to the way it is in a topos. Recall that in a topos the arrow *false* is the characteriser of a unique arrow from the initial object into T. In order to make a similar definition for a quantum topos such as **Vect** one requires the following result.

In a quantum topos with a zero object O, the arrow $O \to T$ is monic because an arrow $h : Z \to T$ factors uniquely into a monic and epic arrow as in $h = me$ and if h is a zero arrow it factors through O and hence the arrow $O \to T$ is monic.

We observe that in a topos [Moe92] any arrow into the initial object is an isomorphism, which forces the initial and terminal to be distinct if one wishes to avoid the trivial category. For the example of **Vect** however, this is not the case. There are many arrows into the terminal zero object and these are clearly not isomorphisms.

Falseness is defined here for a quantum topos with a zero object, such as the category **Vect**.

Definition 6.18 The arrow *false*, or *frue*, is any monic arrow $\chi_{!,!}$ in a characterising square

$$\begin{array}{ccc} O & \xrightarrow{\;!\;} & T \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \mathrm{fr} \equiv \chi_{!,!}} \\ T & \xrightarrow[\mathrm{tr}]{} & \Theta \end{array}$$

Definition 6.19 The *complement* operation is the characteriser of false in

$$\begin{array}{ccc} T & \xrightarrow{\;\mathrm{fr}\;} & \Theta \\ {\scriptstyle 1_T}\downarrow & & \downarrow{\scriptstyle \neg \equiv \chi_{\mathrm{fr},1}} \\ T & \xrightarrow[\mathrm{tr}]{} & \Theta \end{array}$$

Recall that this characteriser is the composition $\varepsilon_\Theta \cdot FGF(\chi_{G(\mathrm{fr})}) \cdot \omega_\Theta$.

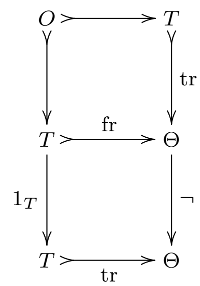
Example 6.20 In the category **Vect**, frue is a second choice of one dimensional subspace $K \rightarrowtail K \oplus K$. The intersection of the two distinct subspaces, true and frue, is the zero vector space. Superposition thus begins to play a significant role in the logic of **Vect** as a quantum topos, as its physical use would suggest it should.

6.9 Properties of Complementation

In this section it is shown that complementation has the properties required for its usual interpretation. As well as $\neg\mathrm{fr} = \mathrm{tr}$ one has

Lemma 6.21. $\neg tr = fr$

Proof. The right–hand side of

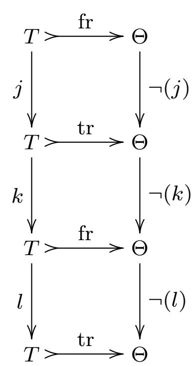


is the definition of frue in this double pullback. □

Allowing for the dependency on scalars, given arrows $j, l \in \mathbf{Bra}(\mathrm{fr})$ and $k \in \mathbf{Bra}(\mathrm{tr})$

$$\neg(lkj) \sim \neg(l) \cdot \neg(k) \cdot \neg(j)$$

This follows from the composition

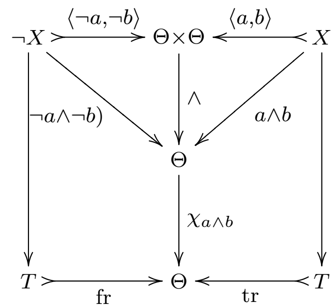


This rule resembles the distinguishing feature of complementarity in intuitionistic logic. Observe that the centre square is not a pullback, so it does not immediately follow that $\neg\neg = 1_T$ as in Boolean logic.

We must also verify that the complement is defined usefully with respect to conjunction.

Lemma 6.22. *For a, b monics into Θ, $\neg(a \wedge b) = \neg a \wedge \neg b$*

Proof. Follows from the diagram



□

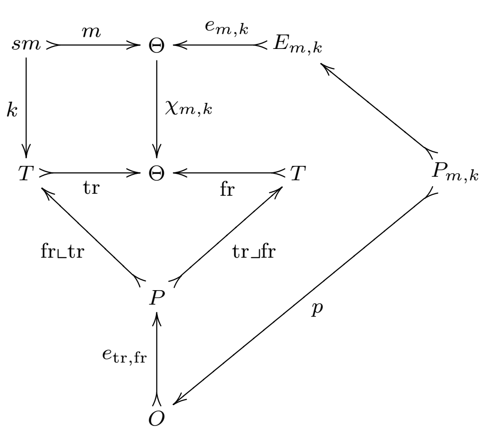
6.10 Quantum Lattices

We will now examine the extension of the notion of a distributive lattice of subobjects in a linear topos. Such a topos has products and equalisers. In this section the existence of an arrow frue is assumed.

Given tr and fr, let O be the equaliser of $P \equiv \mathrm{tr} \wedge \mathrm{fr}$. If O is the initial object of $\Im$ it defines a *bottom* element for the lattice $\mathbf{Sub}(\Theta)$ of two–valued quantum logic. Note however that were we to replace frue by a relative truth tr_i, although O may not be initial, one would still have lattice completion in the following sense.

Proposition 6.23. *Given any monic m into Θ, there exists a state $(\tilde{P}_{m,k}, \tilde{k})$ contained in O which is also contained in $[(m,k)]$ in $\mathbf{Sub}(\Theta)$.*

Proof.



Set $e_{m,k}$ to be the equaliser of $\mathrm{tr} \cdot k$ and $\chi_{m,k}$. Let $P_{m,k}$ be the pullback of $\mathrm{tr} \cdot \mathrm{fr} \llcorner \mathrm{tr} \cdot e_{\mathrm{tr,fr}}$ and $\chi_{m,k} \cdot e_{m,k}$. This provides a monic $p : P_{m,k} \rightarrowtail O$. Let $\tilde{P}_{m,k}$

be the pullback of $P_{m,k} \rightarrowtail E_{m,k}$ composed with $e_{m,k}$ and m. Then $\tilde{P}_{m,k}$ is contained in both sm and O, the latter by factoring through $P_{m,k}$. □

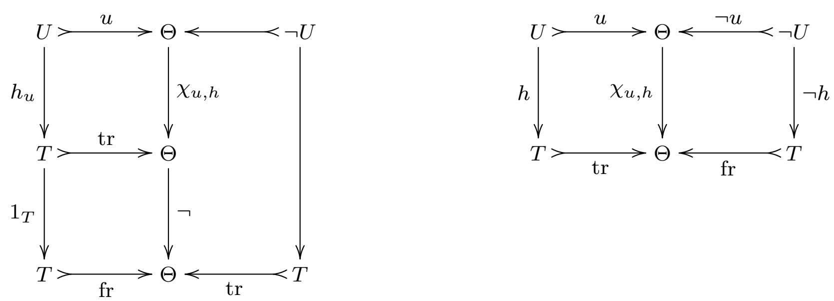
Recall that the lattice of subspaces of a Hilbert space H [Pir76] is *ortho-complementary.* That is, it is equipped with a complement satisfying

1. $U \leq V$ implies $\neg V \leq \neg U$

2. $\neg\neg U = U$

3. $U \vee \neg U = 1 \qquad U \wedge \neg U = 0$

These rules force Boolean logic in a topos. What represents the objects $\neg V$ and $\neg U$ in a quantum topos? The two possible definitions for $\neg U$ in the diagrams

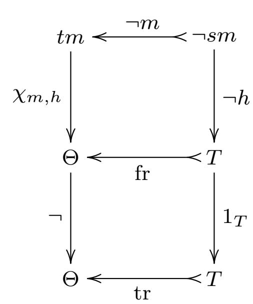


(83)

agree, since the rectangle can be decomposed into two pullbacks. $\neg V$ may be defined similarly.

Lemma 6.24. $\chi_{\neg m, \neg h} \sim \neg \chi_{m,h}$

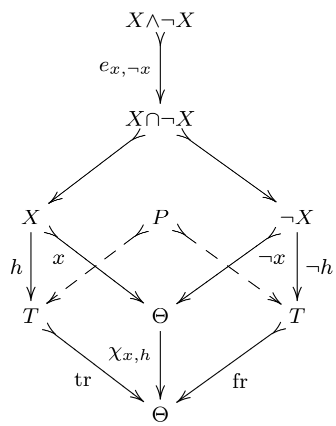
Proof. The left–hand vertical composition of



must be similar to the required arrow by axiom 3. □

Lemma 6.25. $X \wedge \neg X \simeq O$

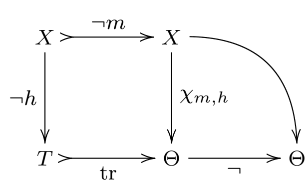
Proof.



By the cube lemma [McL92] there exists an arrow $X \cap \neg X \to P$ such that the cube is a cube of pullbacks. Since $O \rightarrowtail P$ is an equaliser there exists a unique arrow $O \rightarrowtail X \wedge \neg X$ by the universal property, which is monic as the first component of a composition defining the equaliser. Similarly, there is an arrow $X \wedge \neg X \rightarrowtail O$ which sets up an isomorphism as required. □

But negation must actually act on subobjects $[(m, h)] \in \mathbf{Sub}(X)$. It does so via characterisation.

Definition 6.26 $\neg[(m, h)]$ is the pair of arrows defined by the pullback of $\neg\chi_{m,h}$ and $\neg$tr



Proposition 6.27. $\neg[(1_X, h)] = O$ *and* $\neg\neg O = O$.

Proof. Observe that $\neg\chi_{1_X,h} = \mathrm{fr}\cdot h$. Pulling back along tr gives an object which must be isomorphic to the pullback of fr along tr, namely O. Conversely, apply $\neg\neg$ to obtain $\neg\neg O = \neg\neg(\neg 1_X) = \neg 1_X = O$. □

Definition 6.28 $U \leq V$ if $U \wedge \neg V = O$.

Proposition 6.29. *If* $U \leq V$ *then* $\neg V \leq \neg U$.

Proof. Apply $\neg\neg$ to $U \wedge \neg V = O$ to obtain $\neg\neg U \wedge \neg\neg\neg V = \neg\neg U \wedge \neg V = \neg\neg O = O$ as required. □

Thus the orthocomplement property (1) holds. We saw above that the complement automatically satisfies a weak form of property (2). This leaves property (3), one half of which has been shown. The disjunction property may be replaced by the rule

$$\neg U \vee \neg\neg U = \neg\neg 1 \tag{84}$$

which follows easily from the given rule and the definition $\vee = \neg\wedge$. Observe that this reduces to the usual rule when the complement is Boolean. Disjunction will be considered in the following section. In summary, ignoring the possible existence of quantum meets, a new definition for quantum lattice begins with the following.

Definition 6.30 A *semiclassical lattice* $\mathbf{Q}$ is a thin category with special object O and terminal object 1, equipped with a complement $\neg$, a meet $\wedge$ and join $\vee$ such that

1. the $\neg\neg\neg \simeq \neg$ law holds
2. $U \leq V$ implies that $\neg V \leq \neg U$
3. $\neg U \vee \neg\neg U = \neg\neg 1 \qquad U \wedge \neg U = O \qquad$ for all U

Definition 6.31 A *quantum lattice* is a semiclassical lattice equipped with quantum meets $\overline{\wedge}$ and quantum joins $\overline{\vee}$.

This naturally introduces a non–commutative element into the notion of space, associated to a braiding for $\Im$.

For completeness we include here some tentative and simple remarks on topologies on a quantum topos, in this elementary setting. Recall that for a topos a topology is a choice of arrow [McL92] $j : \mathbf{\Theta} \to \mathbf{\Theta}$ obeying certain laws. Double negation $\neg\neg : \Theta \to \Theta$ satisfies this axiom. Observe that for a quantum topos one might use a scaled double negation arrow $j \equiv \neg(k) \cdot \neg(l)$. In that case, the diagram of complement composition requires that

$$k = klk \tag{85}$$

This regularity requirement puts a strong condition on the pairing of complements to form topologies in the classical sense. For **Vect** one naturally has that $k = l^{-1}$. Observe that this is actually like a normalisation condition in the quantum mechanical sense, since it says that localisation with respect to the double negation topology cannot scale the expression $\neg\neg|0\rangle = |0\rangle$. Thus one might argue that there is no need to adjust the definition of a topology, at least for a consideration of the logic of vector spaces.

Of course the notable omission from the classical definition of a topology is the lack of diagrams related to the braided monoidal structure. It seems reasonable to add at least a fourth condition

$$\begin{array}{ccc} \Theta\otimes\Theta & \xrightarrow{\overline{\wedge}} & \Theta \\ {\scriptstyle j\otimes j}\downarrow & & \downarrow{\scriptstyle j} \\ \Theta\otimes\Theta & \xrightarrow[\overline{\wedge}]{} & \Theta \end{array} \tag{86}$$

subject to adjustments involving the braiding.

6.11 Disjunction

In a topos, there are two equivalent notions of disjunction for truth values x and y, namely

1. $x \vee y = \neg(\neg x \wedge \neg y)$
2. $x \overline{\vee} y = \forall_z(((x \to z)\overline{\wedge}(y \to z)) \to z)$

Note that the universal quantifier $\forall_z$ is applied here to generalised truth values, or *formulae*, as it was in the fundamental theorem.

Recall that to say that a formula $\phi : A_1 \otimes A_2 \otimes \cdots \otimes A_n \to \Theta$ is any generalised truth value in $\Im$ is not quite correct. As we saw, this arrow is really an *interpretation* of an abstract formula in terms of vector variables of type $\underline{A}$. To obtain a relation associated to such an interpretation, take the pullback of ϕ with tr

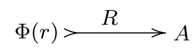
$$\begin{array}{ccc} \Phi(\underline{A}) & \xrightarrow{R} & A_1\otimes A_2\otimes\cdots\otimes A_n \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ T & \xrightarrow[\mathrm{tr}]{} & \Theta \end{array} \tag{87}$$

Consider the case when ϕ is the formula of definition 2, but without the quantifier. First, let z have source A as does x. Consider the arrow $\langle x, z\rangle$:

$A \to \Theta \times \Theta$. Composition of $\langle x, z\rangle$ with $\to_u \cdot!(h)j_{\Theta\Theta}$ defines an arrow $(x \to z) : A \to \Theta$. This is the usual definition in the case of a topos. Now the arrow $(x \to z)\overline{\wedge}_k(y \to z)$ is defined by the composition

$$\overline{\wedge}_k\cdot!(h)j_{\Theta\Theta}\langle(x \to z), (y \to z)\rangle$$

This is an arrow $A \to \Theta$, so the addition of the final $\to z$ yields another arrow $r : A \to \Theta$. By pulling back this interpretation along tr we obtain



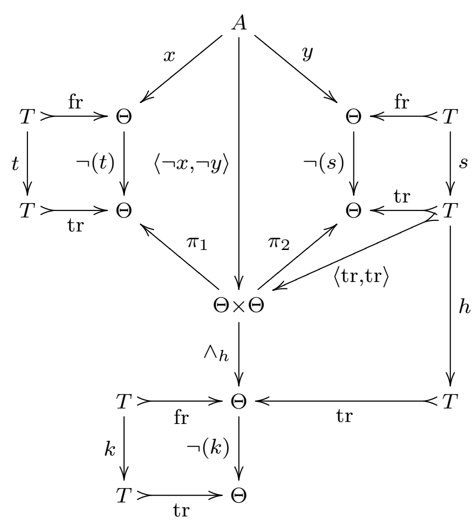
(88)

to which a universal quantifier $\forall$ may be applied.

Now consider the definition 1 above, with respect to the ordinary conjunction.

Definition 6.32 $x \vee y$ is defined by the vertical composition of the diagram

(89)



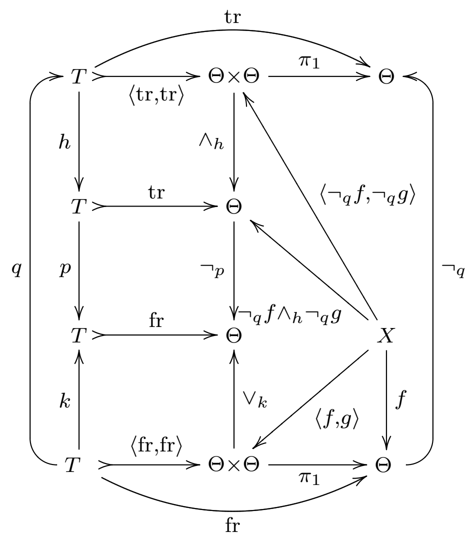
in terms of the four scalars s, t, h and k.

This definition yields the desired rule,

Proposition 6.33. *Given arrows $f, g : X \to \Theta$ and bra state arrows $h, k, p, q : T \to T$ with $k = phq$ then*

$$f \vee_k g = \neg_p(\neg_q f \wedge_h \neg_q g)$$

Proof. The complete diagram



(90)

is commutative and the result sits in the central triangle. □

Continuing with this line of reasoning, existential quantification $\exists_X$ may be defined exactly as $\forall_X$ is defined, but with respect to disjunction rather than conjunction. Although we have not considered at all the adjunctions relating the subobject functor to quantifiers [Moe92], the usual concepts are clearly recovered in the case of a topos.

6.12 Example: Vector Spaces

In summary, it is shown here that the category **Vect** of vector spaces and linear maps over the complex number field $\mathbb{C}$ is an example of a quantum topos.

The adjunction between **Set** and **Vect** works as follows. The functor F takes a set A to the vector space of all formal linear combinations of elements of A, which is therefore a basis for $F(A)$. It sends the monoidal structure of Cartesian product in **Set** to tensor product in **Vect**, and in fact the universal property of Cartesian product is really used to define the tensor product. In the other direction, the functor G simply forgets the vector space structure on an object V, the elements of which become elements of a set $G(V)$. In **Vect**, the counit natural transformation between FG and $1_{\mathbf{Vect}}$ given by the arrow

$FG(V) \to V$ which takes all elements

$$\sum \lambda_v v$$

for set elements v in V to the vector v. Similarly, in **Set** there is a unit natural transformation $1_{\mathbf{Set}} \Rightarrow GF$ which takes elements $a \in A$ to the element $1.a$ as a formal (linear) combination of elements of A. The squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ GFA & \xrightarrow[GFf]{} & GFB \end{array}$$

commute because Ff will take basis elements to basis elements, and G simply forgets the vector space structure, preserving the map f on elements of A.

As stated above, the truth arrow in **Set** is mapped to an arrow $\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ which is thought of as a qubit with the $|1\rangle$ state selected. Note that we have used coproduct here, but the point is simply that $\mathbb{C} \oplus \mathbb{C} \simeq \mathbb{C}^2$ is the image of the two point set under F. We would like to know which monic arrows $V \rightarrowtail W$ lie in pullback squares

$$\begin{array}{ccc} V & \rightarrowtail & W \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \chi} \\ \mathbb{C} & \underset{t}{\rightarrowtail} & \mathbb{C}\oplus\mathbb{C} \end{array}$$

Firstly, consider V to be any subspace of a finite dimensional $W \otimes \mathbb{C} \simeq W$. This requires χ to map all elements of V to the one dimensional state $|1\rangle$, namely the second factor $\mathbb{C}$. Now consider any vector space Q with maps $k : Q \to \mathbb{C}$ and j such that $\chi \cdot j = t \cdot k$. Then it must also be true that χ maps elements of Q in W to the one dimensional space $|1\rangle$. In other words, the map j may be factored through V via $q : Q \to V$. Since t is monic it follows that $k = hq$ as required. This provides a large class of characterisable monics in **Vect**. Moreover, these subspace arrows are the ones traditionally associated with state vectors in ordinary quantum mechanics.

The braiding in **Vect** is given by the basic flip map

$$\gamma_{VW} : V \otimes W \to W \otimes V$$

which sends $v \otimes w$ to $w \otimes v$ for elements $v \in V$ and $w \in W$. Recall that the tensor product [Kas95] of V and W is characterised essentially uniquely by the following property. Let $\mathrm{Hom}(U \times V, W)$ be the set of bilinear maps into a vector space W. Then there exists a map $x : U \times V \to U \otimes V$ such that, given a linear map $f : U \otimes V \to W$, the map $f \cdot x$ from $\mathrm{Hom}(U \otimes V, W)$ to $\mathrm{Hom}(U \times V, W)$ is an isomorphism. By definition, the element $v \otimes w$ is given by $x(v, w)$. It follows that tensor product is the left adjoint of a Hom functor, since

$$\mathrm{Hom}(U \otimes V, W) \simeq \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

Since Cartesian product is associative, there is an associativity isomorphism between $(u \otimes v) \otimes w$ and $u \otimes (v \otimes w)$ given by the map x. Exponential objects are given by the spaces $\mathrm{Hom}(V, W)$, which are linear under the usual rules, such as $\lambda(f + g) = \lambda f + \lambda g$ for a scalar λ. A scalar $h : \mathbb{C} \to \mathbb{C}$ in **Vect** is simply a complex number, taken generically to be non-zero. Thus states are specified by pairs (m, h) for m a monic $\mathbb{C} \rightarrowtail V$ and h a number.

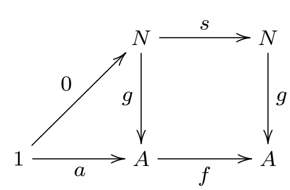
The arrow $\mathrm{tr} \otimes \mathrm{tr}$ from $\mathbb{C}$ to $\mathbb{C}^4$ appears in the definition of monoidal conjunction. Here $\mathbb{C}^4$ is the tensor product of two qubit states, with a natural basis set of four product states in $|0\rangle$ and $|1\rangle$. The arrow $! : \mathbb{C}^4 \to \mathbb{C}^4$ is the isomorphism from this product basis to $\mathbb{C}^2 \times \mathbb{C}^2$ given by $|01\rangle \mapsto (|0\rangle, |1\rangle)$ etc. For $x \otimes y : X \otimes Y \to \mathbb{C}^4$ the arrow $x \overline{\wedge} y : X \otimes Y \to \mathbb{C}^2$ is given by the composition of $x \otimes y$ with the arrow $!$ (which is $\langle \chi_{\mathrm{tr,tr}}, \chi_{\mathrm{tr,tr}} \rangle$ with respect to the product $\mathbb{C}^2 \times \mathbb{C}^2$) and the arrow $\wedge$ defining ordinary intersection of vector spaces. That is, monoidal conjunction simply lifts ordinary conjunction into tensor product spaces. Since a non-trivial intersection of two vector spaces requires that one space lies wholly in the other, the conjunction operation need only project onto this subspace.

If truth is described by the arrow mapping $\mathbb{C}$ onto $|1\rangle$, then falsity may correspond to the embedding $\mathbb{C} \rightarrowtail |0\rangle$. Thus complement on monics expresses a choice of orthogonal subspace, although the concept of orthogonality via inner product is not necessary. The bottom element for lattices is then the zero vector space, which is the intersection of true and false.

Although not distributive for disjunction, lattices of subspaces obey the quantum topos rules. Ordinary disjunction was defined by $x \vee y = \neg(\neg x \wedge \neg y)$ for arrows into the qubit. Consider the example of two distinct 1-dimensional spaces. The intersection of their complements will be the zero space, and the complement of this is the full 2-dimensional space. Thus disjunction does indeed linearise union. The definition also allows one to scale the union by a numerical factor such that complements and conjunction are also scaled accordingly. Monoidal disjunction is similarly defined with respect to monoidal conjunction.

6.13 The Internal Language

Classical categorical logic [Sco86][McL92] is constructed from formal *types* and a given type has countably many variables. Since types are given by the objects of the topos, this suggests working with a topos that has a natural number object, such as the topos **Set**. Rather than imposing further restrictions directly upon the quantum topos, we assume that the classical topos ξ is equipped with a natural number object. This object N comes with a successor arrow $s : N \to N$ which adds one to an ordinal $n \in N$. Thus for any object A and arrow f, there exists an arrow g such that the diagram



commutes. In a quantum topos there is then an arrow $o : F(1) \to F(N)$. In **Vect**, for example, this arrow chooses a one dimensional subspace of the infinite dimensional vector space. We say that o is a *term* of type $F(N)$. A *variable* of type X in the quantum topos is an indeterminate arrow $F(1) \to X$. As discussed above, a *formula* is a term of type Θ. Given a formula $\phi(x)$ there is a term of type Θ^X written $\{x \in X | \phi(x)\}$. There must exist certain formulas. Given ϕ and ψ of type Θ, there are also formulas $\phi \wedge \psi$, $\phi \overline{\wedge} \psi$, $\phi \overline{\vee} \psi$, $\phi \vee \psi$ and $\phi \Rightarrow \psi$. According to the definitions of this chapter, such arrows do indeed exist in the category.

Proofs $L(x_1, x_2, \cdots, x_n) \vdash_X \phi(x)$ will often occur with variables of type

$X_1 \otimes X_2 \otimes X_3 \otimes \cdots X_n$, an object which really requires a choice of bracketing to properly identify, but since this expands the required rules in an obvious way the issue will be ignored here.

Definition 6.34 [Sco86] The *internal language* of a quantum topos $\Im$ has objects as types, terms in variables of each type and special objects $F(1)$, Θ and $F(N)$ along with special terms

1. scalars $h : F(1) \to F(1)$
2. the truth arrow tr of type Θ
3. an arrow $o = F(0) : F(1) \to F(N)$
4. further terms of type $F(N)$ obtained by composition with $F(s)$

and local *entailment* $\vdash_X$ operations on formulas which obeys the rules

1. $\phi \vdash_X \mathrm{tr}$
2. $\phi \vdash_X \phi$
3. if $\phi \vdash_X \psi$ and $\psi \vdash_X \upsilon$ then $\phi \vdash_X \upsilon$
4. $\phi \vdash_X \forall_Y \psi(y)$ means that $\phi \vdash_{X \coprod Y} \psi(y)$
5. if $\phi(x) \vdash_X \psi(x)$ and $\psi(x) \vdash_X \phi(x)$ then $\phi(x)$ and $\psi(x)$ are weakly equal as arrows in $\Im$

Observe that the classical rules of thinning and contraction are omitted, as they are for linear logic. However, for classical objects A in $\Im$ local entailment $\vdash_A$ could be made to satisfy these rules, and this would allow recovery of the ordinary internal logic for a classical topos.

Unconditional entailment $\vdash \phi$ as an internal statement means that the arrow ϕ is equal to the arrow tr, which is an internal concept of truth, but this may never happen except for truth itself. For example, the axiom of Peano stating that 0 has no successor is true in a classical topos: $\vdash \forall_N (sn = 0 \Rightarrow \mathrm{fr})$. What happens to this statement under the functor F? We are now comparing the arrows o and $F(sn)$ in $\Im$. Equality of the arrows must be expressed by an equality of formulas

$$\mathrm{tr} \cdot l \cdot o = \chi_{j_{F(N)F(N)\Delta_{F(N)}}} \cdot j_{F(N)F(N)} \langle o, F(sn) \rangle \tag{91}$$

for a scalar l. This can only imply falsity up to a scalar h via implication. With quantification over $F(N)$ introducing yet another scalar k, the final quantum statement depends on three scalars, but is nonetheless true in this weak sense.

Remark 6.35 At least in **Vect**, the object $F(N)$ may also be considered classical. This opens up the possibility of considering $F(N)$ $(= K^{\infty})$ itself as an alternative basis for weakened truth in the quantum mechanical sense of truth as measurement.

Given any formula $\phi(\underline{x})$ in $\Im$, the *internal semantics* assigns an extension $\Phi(\underline{x})$ to the formula, as defined in section 6.6. An extension represents the collection of variables $\underline{x}$ such that ϕ is (weakly) true.

The logic of a quantum topos, with a weakened notion of truth, behaves a lot like the multiplicative-additive fragment of linear logic, which is based on the concept of resource sensitive proofs rather than a rigid classical truth. In this interpretation the (commutative) tensor product of two objects denotes a simultaneous use of resources, which is now seen to be similar to the quantum mechanical combination of systems. Direct sum in linear logic represents a range of choices, analogous to the range of possible outcomes in the decomposition of a complex physical system.

A morphism between two classical languages [Sco86] is an arrow for variables and terms that preserves $\mathbf{1}$, N, Ω, product and exponentiation. (Note that all these objects have analogues in a linear topos via the adjunction map F). There is an adjunction [Sco86] between this category **Lang** of languages and a restricted category of toposes **Topos** with strict logical morphisms, which are functors that preserve all the elements above and also the map s, the zero number, the initial object $\mathbf{0}$, evaluation arrows, truth, transposition and product structure. The adjunction functor takes a topos ξ to its internal language $L(\xi)$.

It may be unsatisfactory to consider the relation between a category of linear toposes and languages in a one or two dimensional setting, but the existence of adjunctions between objects of **Topos** and linear toposes suggests the possibility of relating quantum logic to type theories via a composition of functors. This would require a natural functor $z : \xi \to \textbf{Topos}$ embedding the topos associated to the quantum topos in the category of all toposes. Then $L \cdot z \cdot G$ would take a quantum topos to an associated internal classical language.

It will be interesting to look at the higher categorical structures needed to establish such an adjunction. Essential ideas will probably come from higher dimensional topos theory, such as the constructions considered by Weber [Web] for categories of categories.

7 Higher Categorical Directions

"Eventually, we will allow bicategories as coefficient objects."

R. Street [Str87]

In this chapter some more speculative connections between higher dimensional structures and causality in physics are discussed. The constructions of the last two chapters were formulated in terms of arrow diagrams, even though monoidal categories have a higher dimensional aspect. This was possible because we were dealing with categories with only two levels of arrows, treated as objects and 1-arrows. A further investigation of quantum toposes for M-theory would require familiarity with higher dimensional structures. Note that categorical dualities have also been neglected in this work, although they are crucial to ribbon structures, or to a definition of a category of Hilbert spaces.

Recall that a monoidal category is precisely a one object bicategory (see appendix). A braided monoidal category is in fact a one object, one 1-arrow tricategory [Str95]. Here we briefly introduce strict cubical tricategories before discussing some of their consequences for physics.

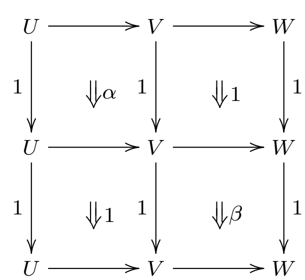
7.1 Strict Cubical Tricategories

Just as bicategories are biequivalent to strict 2-categories, tricategories are triequivalent to strict cubical tricategories. The collection of all these categories is known as **Gray** after the discoverer of the remarkable universal product for bicategories, John W. Gray [Gra74][Gur]. Having defined bicategories in the appendix we will say simply that **Gray** is the 3-dimensional category of bicategories, pseudofunctors, pseudonatural transformations and modifications.

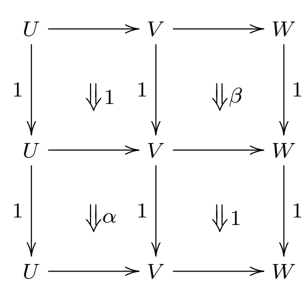
We have freely used Cartesian products for both sets and 1-categories, but in considering the combination of two quantum toposes $\Im_1 \times \Im_2$ we really should take into account that monoidal structures are at least bicategories.

Physically, such a pairing of personal universes is inevitable. Recall that in twistor theory the sheaf of germs of holomorphic functions on a celestial sphere is a representation of a *single classical event*, and the collection of all sheaves on the sphere is an example of a classical topos. A comparison of events in a truly relational setting is viewed as a comparison of local universes, or individual gravitational logics, where an observer must carry a measurement template not just of a single event but of a universe of events. In this thesis, only ordinary quantum logic was discussed, but even in this case the higher dimensional nature of the physical logic seems inescapable. Although this view may be interpreted initially as an extension of MWI beyond the 2-categorical linear realm of ordinary quantum mechanics, it treats local logics, or universes, very differently. In contrast to MWI, it is not necessary to keep track of copies of concrete branching universes, provided the constraints of the observational question may be phrased in the logical language. In other words, the higher topos point of view aims to eliminate background baggage, which is an artefact of a fixed notion of classical spacetime.

The Gray tensor product [Gra74][Gur] takes objects U in $\Im_1$ and V in $\Im_2$ to pairs (U, V). The 1-arrows are strings made out of 1-arrows g_i in $\Im_1$ and f_j in $\Im_2$. That is, formal pairs $(g_i, 1_{V_i})$ and $(1_{U_j}, f_j)$ such that $(g_1, 1_V)(g_2, 1_V) = (g_1 g_2, 1_V)$ and $(1_U, f_1)(1_U, f_2) = (1_U, f_1 f_2)$. For each pair f and g there is an isomorphism $\sigma_{fg} : (1_U, f)(g, 1_V) \Rightarrow (g, 1_V)(1_U, f)$. This means the product has the property that the diagram

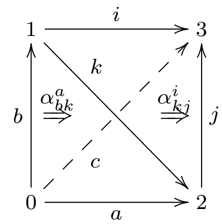


may be different, via an isomorphism, from the diagram



It was Crans [Cra99] who stressed the importance of the dimension raising aspect of the Gray product. We are used to the composition of two 1-arrows giving another 1-arrow, or the horizontal composition of squares in a bicategory resulting in another well–defined square. But here the horizontal composition of two 2-arrows results in a *three* dimensional arrow.

Example 7.1 The $6j$ symbol of the Racah–Wigner calculus [Joy00] is a tetrahedron



with edges labelled by spins, and where only the front faces are labelled here. If one composes the front two faces, and also the rear two faces, of a tetrahedron one is left with a horizontal composition of two 2-arrows

$$(\alpha^a_{bk} \cdot \alpha^i_{kj}) \otimes (\alpha^c_{aj} \cdot \alpha^c_{bi})$$

which naturally defines a 3-arrow C^{ijk}_{abc} between 2-arrows $(1_{ja}(\alpha^c_{aj} \cdot \alpha^c_{bi}))((\alpha^a_{bk} \cdot \alpha^i_{kj})1_{ib})$ and $((\alpha^a_{bk} \cdot \alpha^i_{kj})1_{ja})(1_{ib}(\alpha^c_{aj} \cdot \alpha^c_{bi}))$. Although tetrahedra, such as those appearing in state sum models for three dimensional invariants, are pieces of a braided *monoidal* structure, this property of any tetrahedron hints at the premonoidal nature of $6j$ symbols.

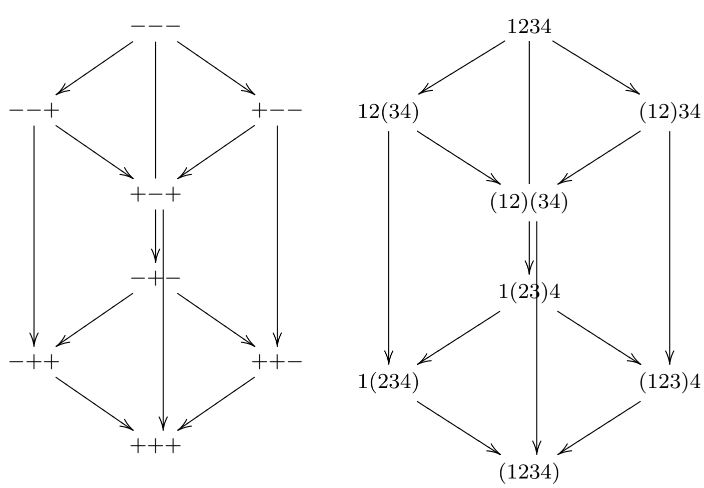
7.2 Tricategories in Physics

The essential presence of weakness in tricategories has direct implications for higher (co)homology, which can no longer be constructed using strict simplicial complexes. The motivic cohomology of chapter 4 is already *weak* in the sense that higher dimensional associahedra define coherence laws for general n-categories. For example, note the appearance of the three dimensional polytope in the axioms for a tricategory. In other words, although only 1-operads were considered, the motivic construction mixes all categorical dimensions in its homological cells. Basically, the real dimension of the underlying space determines the categorical dimension.

Since we search for mass generation exactly in dimension three, special features of tricategories may clarify the distinction between massive invariants and those allowed by the flat space 2-categorical constructions, such as the 2-group representation theory spin foam models currently being developed [Fre] for QFT observables. In fact, these spin foam 2-categories are categories of pseudofunctors [She] for which tricategorical axioms should not be ignored, and thus this is the most natural route for examining extensions of QFT to the gravitational realm.

In a representation category, or in **Vect** for that matter, the objects are state *spaces*, but a particle is specified by a state vector lying within such a space. We attempt to access further information about such vectors by lifting the categorical dimension and looking at structures with vector like objects.

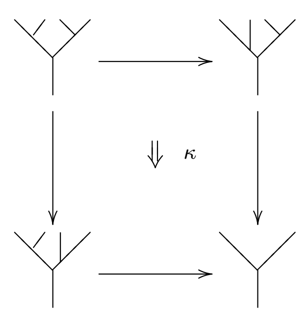
Firstly, we consider lifting monoidal structures up a dimension into tricategories. The edges of the Mac Lane pentagon become five sides of a cube. This so-called parity 3-cube [Strb] and its labellings for tricategorical data looks like



where the numbers now replace the $T(p, q)$. These objects may be loosely thought of as the bicategories of particle states. In this setting, state composition is now via the Gray tensor product, as discussed in [Cra99]. This dimension raising aspect of system composition was also apparent in the language of operads, as it arose in the discussion of gluon amplitudes. In fact, the combinatorics of operads [Bata] are responsible for defining the coherence laws for higher categories, so this is no coincidence. This suggests studying annihilation and creation operators as (de)categorification processes. At this level of abstraction, the vacuum need no longer be described by a specific state in a fixed Hilbert space.

Returning to the representation theory of the QCD Lie group $SU(3)$, and in respecting a quark colour grading for confinement, the category of representations was shown to exhibit a broken pentagon structure in [Joy04][Joy03]. This means that the causal levels (vertices) on trees must always be separated vertically, resulting in trees labelled by permutations rather than associations [Bata]. The broken pentagon is exactly the six sided parity cube, so tricategorical structures naturally impose such physical conditions. This reduction of the associahedra polytopes to the permutohedra for permutations may be carried out using the realisations of Loday [Lod04], which truncate the associahedra by extra hyperplanes. For example, in two dimensions the permutohedron is the familiar hexagon, labelled by elements of the permutation group S_3 on three letters. This group appears as the centre of $SU(3)$ in the colour grading analysis. As observed in chapter 2, the hexagon is a 2-operad polytope corresponding to a 2-level tree with three leaves.

Although the view is that Lie symmetry is not itself fundamental to an understanding of the Standard Model, by observing that the objects in a category such as $\mathbf{Rep_{SU(3)}}$ are representation spaces rather than particle states, it follows that to capture the notion of a state in category theoretic terms it is necessary to internalise this picture and replace ψ by its tricategorical analogue. For the pentagon this leads to the symmetry breaking of the parity cube, because the top face of the cube cannot be ignored [Strb]. The new top face of the cube is κ, the deformation parameter of [Joy03][Bata]. Its defining square is



(92)

This square appears as a piece of data [Str95] for χ for trimorphisms between tricategories, so that it naturally appears in 4-dimensional structures. Because the interior of the cube gives the 3-arrow π, its existence relies on the differentiation of κ from the composition of the other faces of the cube. Now κ

becomes a quantisation parameter, such as that considered by Kashaev [Kas94] in relation to the $6j$ symbol.

The asymptotics of the $6j$ symbol [Rob95] is known to yield hyperbolic volumes for ideal tetrahedra. The definition of the $6j$ symbol comes from the quantum dilogarithm [Kas94] relation, which as we have seen is really the Mac Lane pentagon for weak associativity. In Kashaev's study of the dilogarithm, broken pentagon relations occurred naturally.

In a different direction, our understanding of Feynman QFT renormalisation has been greatly improved in recent years by Connes, Marcolli, Kreimer [Mar][Kre00] and others. These techniques also relate knot theory and QFT diagrams to multiple polylogarithms and MZVs, but from a perspective which ignores gravitational questions. In fact, the appearance of only integral arguments for MZVs in chapter 4 is viewed as a consequence of the restriction to 1-categories. Recall that the argument increases with the 1-ordinal n and the dimension of the moduli space. In the operad setting it is clear how to generalise MZV type algebras to higher dimensions, and to richer zeta algebras, necessary for describing more general particle types.

Another advantage of the categorical view is the ability to abstract geometry away from fixed number fields and background spaces. The concrete removal of classical base spaces is beautifully captured by the idea of the Big Zariski topos [ed.76]. This category $\mathcal{Z}$ classifies commutative spaces (affine schemes) by replacing them by arrows from the (opposite of the) category of finitely generated commutative algebras, as in the commutative diagram

$$\begin{array}{ccc} & & \mathbf{Alg}^{\mathrm{op}} \\ & \swarrow^{s} & \downarrow{\scriptstyle y} \\ \mathcal{Z} & \xrightarrow[i]{} & \mathbf{Set}^{\mathbf{Alg}} \end{array} \tag{93}$$

where y denotes the Yoneda embedding and i an inclusion. The functor s is the algebro-geometric operation of taking the spectrum and $\mathbf{Set}^{\mathbf{Alg}}$ is the classifying topos for commutative algebras [Moe92].

What is the non–commutative analogue of this idea? From the ribbon graph M-theory point of view, a major motivation in the study of quantum toposes is the possibility of describing non–commutative spaces in this way.

As the fundamental theorem highlighted, there is the question of extensions to Stone duality. Phase space (Pontrjagin) duality or, more generally, lattice duality was considered in [Pra] as the mechanism underlying a Heisenberg uncertainty principle. Duality for not necessarily distributive bounded lattices was considered in [Har], where it was shown that the axiom of choice should be dropped. Note that for Pontrjagin duality [Moe92] the group $U(1)$ plays a special role and is known as a schizophrenic object. The example of projective representation categories suggests that perhaps quantum forms of $SL(2, K)$ should play an analogous role for quantum toposes. This idea is strengthened by the importance of the modular group to number theory and monstrous moonshine operads, both of which have strong connections to recent developments in M Theory.

8 Conclusions

"The classical physicist's expectation, far from being trivial, is wrong."

E. Schrödinger [Sch45]

In this thesis, a new approach to rigorous QFT is introduced, motivated by twistor causality and the remarkable relational mathematics of category theory, the natural language in which to discuss quantum gravitational physics.

The categorical method for the exact calculation of amplitudes, introduced here, generalises the cases considered by Witten [Wit04a] using multiple copies of $\mathbb{RP}^1$. Further work would extend the method to higher operads for the computation of different physical amplitudes. These methods fit into a background independent categorical approach to ribbon graph M-theory [Wal03], which is associated to a preon approach with a mass gap [Bra][PFt]. The preon approach can eliminate both standard SUSY partners and possibly also the need for a conventional Higgs boson. We therefore do not expect that spurious SUSY partners and other such entities will be observed at the LHC.

Operads and categorical logic are the natural language with which to describe this theory, which we hope to expand upon before the advent of data collection from the LHC. Extensions to higher loops would naturally involve the modular operads of Getzler and Kapranov [Kap] which were designed specifically for this purpose.

The lepton mass matrices of [Bra] are also expected to fit into the higher operad framework, since idempotents in Jordan algebras [Rio] are naturally associated with projective geometry. Moreover, the idempotent method of computing the number of generations agrees with our Euler characteristic counting for the moduli of the six punctured sphere. It is expected that the 3×3 matrices of the octonion exceptional Jordan algebra will play an important role in understanding these computations, and in investigating related approaches to black hole quantum computation.

It has been shown by Bilson-Thompson [BT] that triple ribbon graphs characterise the basic leptons and gauge bosons of the Standard Model. We would also like to outline more precisely how these diagrams arise in the higher operad context.

Also in this thesis, axioms for a quantum topos were given, and the resultant logic analysed using purely categorical techniques. A simple application of the adjunction with a topos was shown to yield the linear logic of vector spaces. Although we have not discussed additional structure, such a framework might also be applied to tortile tensor categories [Shu94] or other categories with duals.

The utility of categorical structures in clarifying QCD hopefully motivates a more axiomatic approach in general towards the problem of reconciling quantum physics and spacetime geometry. In order to make all of these ideas more precise it will be necessary to study the universal cohomology in the context of quantum toposes, and to set up the combinatorial framework in the correct higher dimensional language [Bae][Bata].

Recently, in [Wit], Langlands duality has been formulated within the framework of four dimensional topological field theories. In [Kap95], Kapranov highlighted the categorical nature of this duality. It would be instructive to draw further links between the operad calculus and this more familiar field theory language.

In summary, this thesis develops a new concept of observable for the Standard Model, and shows that Veneziano functions may be produced via such methods. This thesis also gives an axiomatic definition of quantum topos, and develops its logic to the point of demonstrating that vector space linear logic is recovered.

A Bicategories and Tricategories

A.1 Bicategories

All bicategories are *strictifiable* [Str95] in the sense that they are weakly equivalent to a 2-category in which the special arrows in the following definition are identities.

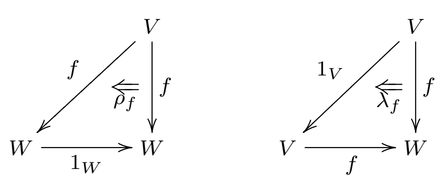
Definition A.1 A *bicategory* is a 2-dimensional category $\mathcal{B}$ containing

1. objects U, V, W, $\cdots$

2. for each object U a pseudoidentity 1_U

3. for each pair of objects U and V, the 1-arrows and 2-arrows of $\mathcal{B}(U, V)$ form a category

4. for each 1-arrow $f : V \to W$ a left identity λ_f and right identity ρ_f

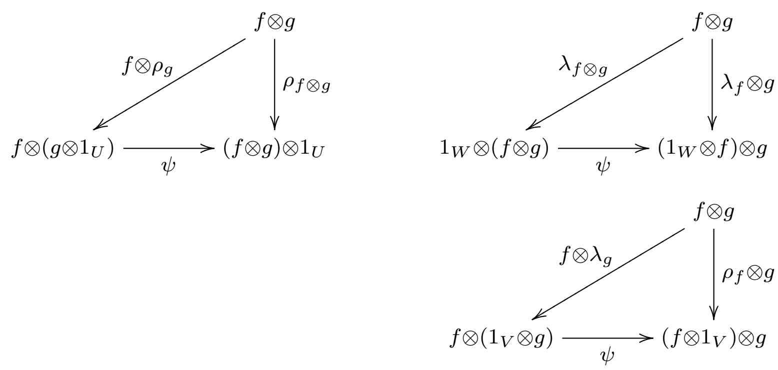
   which are both natural

5. for each triple U, V and W a composition functor

$$\otimes : \mathcal{B}(V, W) \times \mathcal{B}(U, V) \to \mathcal{B}(U, W)$$

6. Associator 2-arrows ψ such that for each quadruple of objects the Mac Lane pentagon holds

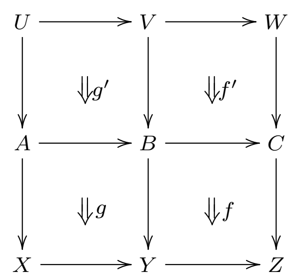
7. commuting triangles



8. the *interchange law* for 2-arrows f, g, f' and g'

$$(f \otimes g)(f' \otimes g') = ff' \otimes gg' \tag{94}$$

The interchange law simply states that a block of four squares



must clearly define a 2-arrow.

Example A.2 The trivial bicategory **1**, with one object, one 1-arrow and one 2-arrow.

Example A.3 Stepping back to the very basic notion of a vector as an arrow, let V be the thin 1-category whose objects are the n-tuples $(c_1, c_2, \cdots, c_n)$ of affine space $\mathbb{A}^n$ and whose arrows are the vectors between these points. Composition of arrows $v+w$ is obvious. A functor between two such categories must satisfy the law of linearity $F(v+w) = F(v) + F(w)$. Commutativity of addition is given by natural transformations. The collection of such categories, functors and natural transformations defines a bicategory of affine spaces and maps.

There are several types of arrow between bicategories. In particular, a *pseudofunctor* $\mathbf{F} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ consists of

1. for an object (U, V) an object $F(U, V)$

2. for every pair (U, V), (Y, Z) a functor

$$F : \mathcal{B} \times \mathcal{B}((U, V), (Y, Z)) \to \mathcal{B}(F(U, V), F(Y, Z))$$

3. for every object (U, V) a 2-isomorphism

$$I_{(U,V)} : F(1_{(U,V)}) \Rightarrow 1_{F((U,V))}$$

4. for every pair $(f, g) : (U, V) \to (Y, Z)$ and $(h, k) : (Y, Z) \to (X, W)$ a 2-isomorphism

$$\phi_{(f,g)(h,k)} : F(h, k)F(f, g) \Rightarrow F((h, k) \otimes (f, g))$$

such that for any 2-arrows τ and σ (with appropriate sources and targets)

$$\begin{array}{ccc} F(h,k)F(f,g) & \xrightarrow{\phi} & F((h,k)\otimes(f,g)) \\ {\scriptstyle F(\sigma)F(\tau)}\downarrow & & \downarrow{\scriptstyle F(\sigma\otimes\tau)} \\ F(\tilde{h},\tilde{k})F(\tilde{f},\tilde{g}) & \xrightarrow{\phi} & F((\tilde{h},\tilde{k})\otimes(\tilde{f},\tilde{g})) \end{array}$$

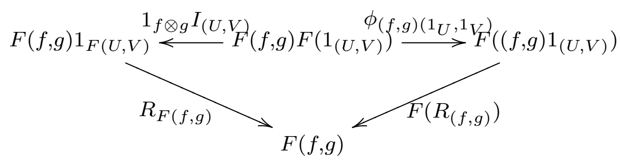
commutes

along with hexagon and triangle relations

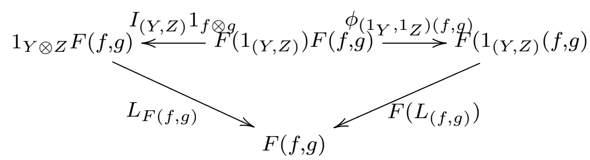
5. for every $(f, g) : (U, V) \to (Y, Z)$, $(h, k) : (Y, Z) \to (X, W)$ and $(r, s) : (X, W) \to (S, T)$

$$\begin{array}{ccccc} F(r,s)(F(h,k)F(f,g)) & \xrightarrow{1_{F(r,s)}\phi_{(h,k)(f,g)}} & F(r,s)F((h,k)\otimes(f,g)) & \xrightarrow{\phi_{(r,s)((h,k)\otimes(f,g))}} & F((r,s)\otimes((h,k)\otimes(f,g))) \\ {\scriptstyle \psi_{r\otimes s,h\otimes k,f\otimes g}}\downarrow & & & & \downarrow{\scriptstyle F(\psi_{(r,s)(h,k)(f,g)})} \\ (F(r,s)F(h,k))F(f,g) & \xrightarrow[\phi_{(r,s)(h,k)}1_{F(f,g)}]{} & F((r,s)\otimes(h,k))F(f,g) & \xrightarrow[\phi_{((r,s)\otimes(h,k))(f,g)}]{} & F(((r,s)\otimes(h,k))\otimes(f,g)) \end{array}$$

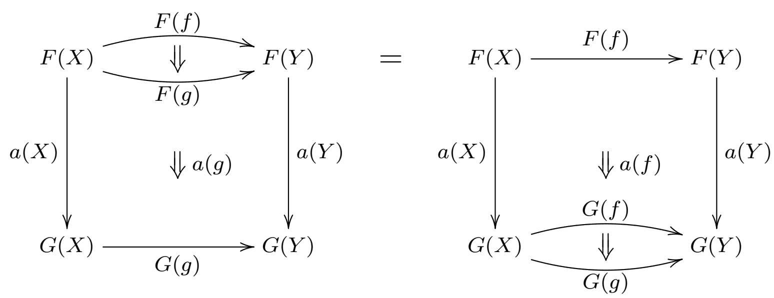
6.



7.

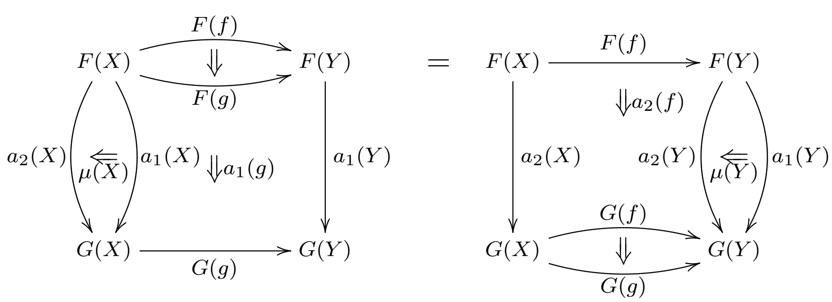


A *2-functor* is a strict 2-dimensional functor between bicategories. Pseudonatural (invertible) transformations a between 2-functors F and G satisfy



(95)

and modifications $\mu : a_1 \to a_2$ between these satisfy the rules



(96)

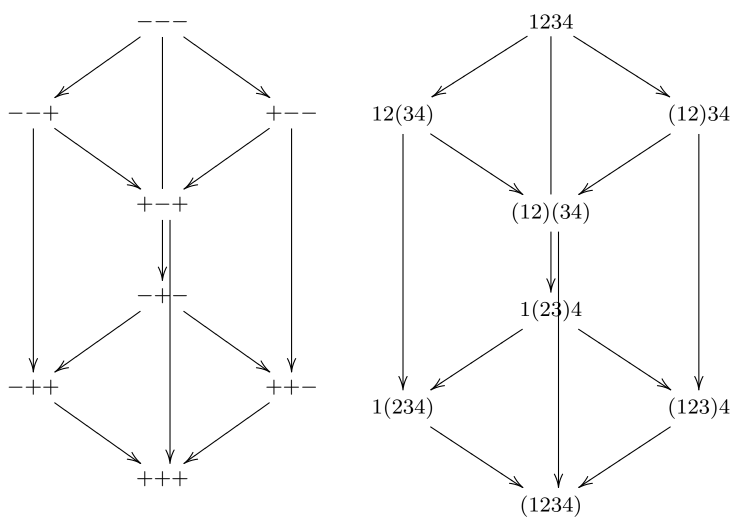
Example A.4 In the sense of representation theory, the 2-Hilbert spaces of [Bae97] are a natural target for 2-functors from a 2-group $\mathbf{G}$, which is a 2-category on one object with invertible arrows in dimension one and two.

A.2 Tricategories

Our notation for tricategories follows [Str95]. We refrain from repeating the rather lengthy full definition, the enormity of which has unfortunately discouraged many from working with higher dimensional algebra. The objects of a tricategory T are labelled p, q etc. For each pair of objects p and q there is a bicategory $T(p,q)$. The internalisation of ψ is a pseudonatural transformation a

$$\begin{array}{ccc} T(r,s)\times T(q,r)\times T(p,q) & \xrightarrow{\otimes\times 1} & T(q,s)\times T(p,q) \\ {\scriptstyle 1\times\otimes}\big\downarrow & \Downarrow a & \big\downarrow{\scriptstyle \otimes} \\ T(r,s)\times T(p,r) & \xrightarrow[\otimes]{} & T(p,s) \end{array} \tag{97}$$

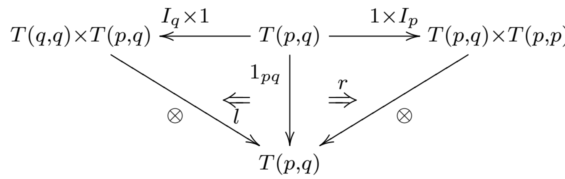
The edges of the Mac Lane pentagon becomes five sides of a cube. This so-called parity 3-cube [Strb] and its labellings for tricategorical data looks like



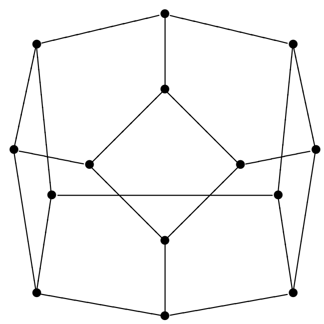
where the numbers now replace the various $T(p,q)$. Note the free use of lower dimensional associativity in choosing bracketings. The other data for a tricategory are

1. for objects p and q, homomorphisms $I_p : \mathbf{1} \to T(p,p)$ for the trivial

bicategory **1** such that there exist pseudonatural equivalences

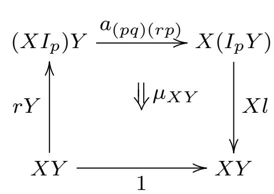


2. a modification π filling in the parity cube, subject to the 4-cocycle Stasheff polytope [Str87]



whose faces represent Mac Lane pentagons for four objects

3. for objects p, q and r the invertible modification pyramid with pieces, for (X, Y) in $T(p,q) \times T(r,p)$



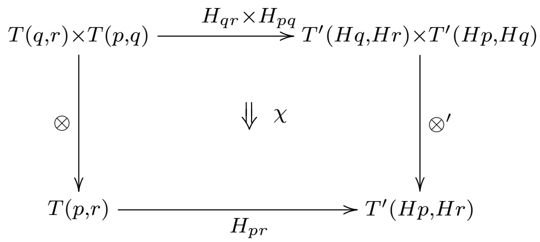
such that appropriate normalisation conditions hold.

A trimorphism $H : T \to T'$ by definition consists of

1. an object function H

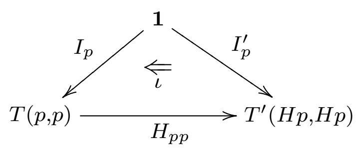
2. for objects p and q of T a pseudofunctor $H_{pq} : T(p,q) \to T'(Hp, Hq)$

3. for objects p, q and r a pseudonatural transformation

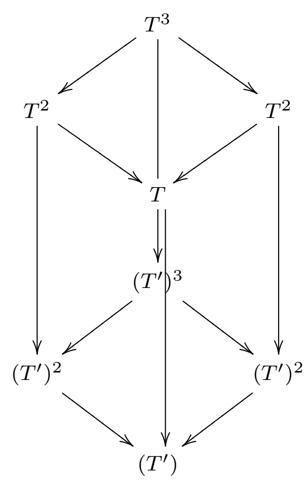


Observe that it is part of the data for χ that is responsible for the non-triviality of the sixth side of the parity cube in a tetracategorical context.

4. pseudonatural transformations



5. for p, q, r, s an invertible modification parity cube ω



Representability for a tricategory is fundamentally different from the one and two dimensional cases. The Yoneda embedding takes the form $\mathbf{Tricat}(T^{\mathrm{op}}, \mathrm{Prep}(T))$ [Str95] where $\mathrm{Prep}(T)$ is a strict, cubical tricategory when T is a cubical tricategory.

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